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Differentiability and ball-covering property in Banach spaces ☆

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ABSTRACT

In this paper, author proves that if X_1 and X_2 are Gâteaux differentiable space, then X_1 and X_2 have the ball-covering property if and only if $(X_1 \times X_2, \|\cdot\|_p)$ and $(X_1 \times X_2, \|\cdot\|_\infty)$ have the ball-covering property, where $\|(x,y)\|_p = (\|x\|_1^p + \|y\|_2^p)^{\frac{1}{p}}$, $p \in [1, +\infty)$ and $\|(x,y)\|_\infty = \max\{\|x\|, \|y\|\}$.

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space. By $x_n \xrightarrow{w} x$ we denote that $\{x_n\}_{n=1}^{\infty}$ is weakly convergent to x. $\overline{C}(\overline{C^w})$ denotes closed hull of C (weak closed hull) and $\mathrm{dist}(x, C)$ denotes the distance of x and C. Let N, R and R^+ denote the set natural number, reals and nonnegative reals, respectively.

Let D be a nonempty open convex subset of X and f a real-valued continuous convex function on D. Recall that f is said to be Gateaux differentiable at the point x in D if the limit

$$df(x)(y) = \lim_{t \to 0} \frac{f(x+ty) - f(x)}{t} \qquad (*)$$

exists for all $y \in X$. When this is the case, the limit is a continuous linear function of y, denoted by $df(x)$. If the difference quotient in $(*)$ converges to $df(x)(y)$ uniformly for y in the unit ball, then f is said to be Frechet differentiable at x. X is called a weak Asplund space [Asplund space] or said to have the weak Asplund property if for every f and D as above, f is "generically" Gâteaux [Frechet] differentiable, that is, there exists a dense G_δ subset G of D such that f is Gâteaux [Frechet] differentiable at each point of G. X is called a Gâteaux differentiability space if every convex continuous function on it is Gâteaux differentiable at the points of a dense set. In 1933, Mazur proved that separable Banach spaces have the weak Asplund property (see [15]). Moreover, it is well known that if X is an Asplund space if and only if X^* has the

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Radon–Nikodym property, X is a Gâteaux differentiable space if and only if every weak* compact convex subset of X^* is the weak* closed convex hull of its weak* exposed points. It is easy to see that

$$\text{Asplund space} \Rightarrow \text{weak Asplund space} \Rightarrow \text{Gâteaux differentiable space.}$$

It is well known there exists a weak Asplund space, but not Asplund space, for example, l^1 is a weak Asplund space, but not Asplund space. In 2006, Waren B. Moors and Sivajah Somasundaram proved that there exists a Gâteaux differentiable space that is not a weak Asplund space (see [14]). In 2002, L. Cheng and M. Fabian proved that the product of a Gâteaux space and a separable space is a Gâteaux differentiable spaces (see [4]).

The study of geometric and topological properties of unit balls of Banach spaces plays a central rule in the geometry of Banach spaces. Almost all properties of Banach spaces, such as convexity, smoothness, reflexivity and the Radon–Nikodym property, can be viewed as properties of the unit ball. We should mention here that there are many topics studying the behaviour of collections of balls. For example, the Mazur intersection property, the packing sphere problem of unit balls, the measures of non-compactness, and the ball topology have all received a great deal of attention by many mathematicians.

Starting with a different viewpoint, a notion of ball-covering property is introduced by Cheng [1]:

Definition 1. A Banach space is said to have the ball-covering property if its unit sphere can be contained in the union of countably many open balls that do not contain the origin. In this case, we also say that the norm has the ball-covering property.

In [2], Cheng proved that if X is a locally uniformly convex space and $B(X^*)$ is w^*-separable, then X has the ball-covering property. In [7], it was established that for every $\varepsilon > 0$ every Banach space with a w^*-separable dual has an $1+\varepsilon$-equivalent norm with the ball-covering property. Clearly, every separable space has ball-covering property, but the converse version is not true. For example, ℓ^∞ is not a separable space, but ℓ^∞ has the ball-covering property (see [1]). In [18], Shang and Cui proved that if a separable space X has the Radon–Nikodym property, then X^* has the ball-covering property. As a corollary, Shang and Cui proved that there exists a non-separable Orlicz function space L_M such that L_M has the ball-covering property. In [3], Cheng and Liu proved that by constructing the equivalent norms on ℓ^∞, there exists a Banach space $(\ell^\infty, \|\cdot\|^0)$ such that $(\ell^\infty, \|\cdot\|^0)$ does not possess the ball covering property. In [19], Shang and Cui proved that if X is separable, X is locally 2-uniformly convex and X is uniformly nonsquare, then there exists a sequence $\{x_n\}_{n=1}^\infty$ of strongly extreme points such that $\cup_{n=1}^\infty B(x_n, r_n)$ is a ball-covering of X, where $\sup_{n\geq 1}\{r_n\} < 1$. The paper is organized as follows. In Section 1, some necessary definitions and notations are collected. In Section 2, author proves that if X_1 and X_2 are Gâteaux differentiable space, then X_1 and X_2 have the ball-covering property if and only if $(X_1 \times X_2, \|\cdot\|_p)$ and $(X_1 \times X_2, \|\cdot\|_\infty)$ have the ball-covering property, where $\|(x,y)\|_p = (\|x\|_1^p + \|y\|_2^p)^{\frac{1}{p}}$, $p \in [1, +\infty)$ and $\|(x,y)\|_\infty = \max\{\|x\|, \|y\|\}$. The topic of this paper is related to the topic of [5,6,8–13,16,17,20,21]. First let us recall some definitions that will be used in the further part of this paper.

Definition 2. A point $x \in S(X)$ is called a smooth point if it has an unique supporting functional f_x. The set of all smooth points of X is denoted by $SmoX$. If every $x \in S(X)$ is a smooth point, then X is called smooth.

It is well known that if $x_0 \in S(X)$ is a smooth point, then convex function $f(x) = \|x\|$ is Gâteaux differentiable at x_0.

Definition 3. A point $x_0^* \in C^*$ is said to be weak* exposed point of C^* if there exists $x \in S(X)$ such that $x_0^*(x) > x^*(x)$ whenever $x^* \in C^* \setminus \{x_0^*\}$.