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# On conjugacies between asymmetric Bernoulli shifts ☆

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ABSTRACT

This paper investigates conjugacies between asymmetric Bernoulli shifts. It is shown that there exist a unique increasing conjugacy and a unique decreasing conjugacy. We respectively construct a sequence of functions to approximate these two conjugacies, and give an estimation for the error of the approximation. We also present explicit formulae of these two conjugacies. It is shown that these two conjugacies are singular, Hölder continuous and not differentiable.



## 1. Introduction

One of the central questions in iteration theory or dynamical systems is to decide whether two self-maps $f : I \to I$ and $g : J \to J$ are topologically conjugate, i.e., whether there exists a homeomorphism $\varphi : I \to J$ such that $\varphi \circ f = g \circ \varphi$. Such a homeomorphism $\varphi$ is called a topological conjugacy or *conjugacy* from $f$ to $g$. In dynamical systems, topological conjugation defines an equivalence relation, which is useful in topological classification of systems.

The conjugacy problem first arose from linearization problem. There are many classic results, for instance, Denjoy's theory [7,8,16], Herman's theory [14,19] for circle diffeomorphisms conjugate to a rigid rotation; Grobman–Hartman theorem [27] for topological linearization of local dynamical systems; Parry's theory [32,25,26] and Milnor–Thurston's theory [22] respectively for continuous piecewise monotone interval maps conjugate or semi-conjugate to a piecewise linear map. There are few works [1,4,5] on the conjugacy problem for piecewise continuous piecewise monotone interval maps. A Lorenz map, considered by Glendinning [13], is a piecewise continuous piecewise monotone interval map. He gave necessary and sufficient conditions for

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a Lorenz map to be conjugate to a linear mod one transformation, and the regularity of some conjugacies is recently investigated in [6].

Consider the asymmetric Bernoulli shift $B_a : [0,1] \to [0,1]$ with a parameter $0 < a < 1$, defined by

$$B_a = \begin{cases} \frac{x}{a}, & 0 \le x \le a, \\ \frac{x-a}{1-a}, & a < x \le 1, \end{cases}$$

which is a simple piecewise continuous piecewise monotone interval map, but different from the class of Lorenz maps in [6]. Specially, if $a = 1/2$, then $B_{1/2}$ is the Bernoulli Shift, also known as *doubling map*. Palmore [23] proved that $B_a$ is chaotic for $0 < a < 1$.

In this paper, we study these conjugacies from $B_{a_1}$ to $B_{a_2}$ where $a_1, a_2 \in (0,1)$ and $a_1 \ne a_2$. The existence and uniqueness are proved for increasing and decreasing conjugacies in the next section. In Section 3, we respectively construct a sequence of functions to approximate the increasing and decreasing conjugacies, and give an estimation for the error of the approximation. Section 4 presents explicit formulae of these two conjugacies. In Section 5, we show that these conjugacies are singular, Hölder continuous, but not differentiable. In addition, we calculate the arc-length of a conjugacy curve and the area under a conjugacy curve.

## 2. Existence of conjugacies

First, we will give two necessary conditions of topological conjugation between $B_{a_1}$ and $B_{a_2}$.

**Lemma 2.1.** *If there exists an increasing conjugacy $\varphi$ from $B_{a_1}$ to $B_{a_2}$, then:*

(i) $\varphi(0) = 0$, $\varphi(a_1) = a_2$, *and* $\varphi(1) = 1$*;*
(ii) $\varphi$ *is a solution of a system of functional equations*

$$\varphi(x) = \begin{cases} a_2\varphi(\frac{x}{a_1}), & 0 \le x \le a_1, \\ (1-a_2)\varphi\left(\frac{x-a_1}{1-a_1}\right) + a_2, & a_1 < x \le 1. \end{cases} \tag{2.1}$$

**Proof.** (i) Substituting $x = 0$ into the equation $\varphi \circ f(x) = g \circ \varphi(x)$, we have $\varphi(0) = \varphi \circ f(0) = g \circ \varphi(0)$. Then $\varphi(0)$ is a fixed point of $g$. So $\varphi(0) = 0$ or 1. Since $\varphi$ is strictly increasing, we have $\varphi(0) = 0$. Similarly, one has the fact $\varphi(1) = 1$.

Now we prove that $\varphi(a_1) = a_2$ by contradiction. Assume that there exists a point $x_0 \ne a_1$ such that $\varphi(x_0) = a_2$. Then there is a small enough neighborhood of $x_0$, denoted by $O(x_0, \delta)$, such that $a_1 \notin O(x_0, \delta)$. On the one hand, it is clear that $\varphi \circ f(x)$ is continuous on $O(x_0, \delta)$. On the other hand, one can see that $g \circ \varphi(x)$ is not continuous on $O(x_0, \delta)$ since $a_2 \in \varphi(O(x_0, \delta))$. This contradicts the equation $\varphi \circ f(x) = g \circ \varphi(x)$. Therefore $\varphi(a_1) = a_2$.

(ii) If $\varphi$ is an increasing conjugacy from $B_{a_1}$ to $B_{a_2}$, then $\varphi([0, a_1]) = [0, a_2]$ and $\varphi([a_1, 1]) = [a_2, 1]$. Thus

$$\varphi\left(\frac{x}{a_1}\right) = \frac{\varphi(x)}{a_2}, \quad \text{if} \quad 0 \le x \le a_1,$$
$$\varphi\left(\frac{x-a_1}{1-a_1}\right) = \frac{\varphi(x) - a_2}{1-a_2}, \quad \text{if} \quad a_1 < x \le 1.$$

Consequently, $\varphi$ is a solution of a system (2.1). □

Remark that when $a_1 = 1/2$ and $a_2 = a$, the solution of the system (2.1) is de Rham's function, which is extensively investigated by many authors, i.e., [2,3,9,20].