J. Math. Anal. Appl. 434 (2016) 271–314

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Twisted second moments of the Riemann zeta-function and applications

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ARTICLE INFO

Article history:
Received 26 May 2015
Available online 14 September 2015
Submitted by B.C. Berndt

Keywords:
Riemann zeta-function
Twisted second moment
Mollifier
Zeros on the critical line

ABSTRACT

In order to compute a twisted second moment of the Riemann zeta-function, two different mollifiers, each being a combination of two different Dirichlet polynomials, were introduced separately by Bui, Conrey, and Young, and by Feng. In this article we introduce a mollifier which is a combination of four Dirichlet polynomials of different shapes. We provide an asymptotic result for the twisted second moment of $\zeta(s)$ for such choice of mollifier. A small increment on the percentage of zeros of the Riemann zeta-function on the critical line is given as an application of our results.

1. Introduction

In [1], Balasubramanian, Conrey and Heath-Brown computed the twisted second moment of the Riemann zeta-function

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\overline{\psi}\psi(\tfrac{1}{2}+it)dt \tag{1.1}$$

where ψ is a Dirichlet polynomial of the type

$$\psi(s) = \sum_{n\le T^\theta} \frac{a(n)}{n^s}, \tag{1.2}$$

and $a(n) \ll_\varepsilon n^\varepsilon$. The length T^θ of the polynomial is sensitive to the nature of the coefficients $a(n)$. They also obtained an explicit main term in their theorem for a particular choice of $\psi(s)$.

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http://dx.doi.org/10.1016/j.jmaa.2015.08.064

In [5], in order to obtain a higher percentage of zeros of the Riemann zeta-function on the critical line, Conrey needed to establish such type of second moment. In his result he made an ingenious choice of $a(n)$ which allowed him to push the value of θ from 1/2 (see [10]) to 4/7. The possibility of obtaining a mollifier by combining two Dirichlet polynomials of different shape had been considered by Lou [11]. In [2], Bui, Conrey, and Young extended (1.1) with an explicit main term for a more sophisticated choice of $a(n)$. They considered $\psi(s)$ as a convex combination of two Dirichlet polynomials of different shape. Introducing such two-piece mollifier increases the complexity and technicality of the computation of the main term. Another such two-piece mollifier was introduced by Feng [9] and the main term was computed explicitly.

Crucial ingredients to obtaining the error term in [9] were Lemmas 1 and 2. To reach $\theta_1 < 4/7 - \varepsilon$ in [5], it was required that $a(n) = \mu(n)F(n)$, for a smooth function F. In [9], the coefficient $a(n)$ in the mollifier was not of the form $\mu(n)F(n)$, for some smooth function F, and it is not clear how the techniques of [5] can be directly applied to the proofs of Lemmas 1 and 2 of [9].

Independently of each other, in [2] and [9], the possibility of obtaining a $\psi(s)$ by combining these three Dirichlet polynomials of different shape was mentioned. One can obtain the main term of (1.1) for such choice of $\psi(s)$ by going over some subtle technicalities in the calculations.

In the present paper we introduce a new mollifier $\psi(s)$ which is a convex combination of four Dirichlet polynomials of different shape. Let

$$\chi(s) := \pi^{s-1/2}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right).$$

We will use the convention

$$P_i[n] := P_i\left(\frac{\log(y_i/n)}{\log y_i}\right) \quad \text{and} \quad \tilde{P}_k[n] := \tilde{P}_k\left(\frac{\log(y_4/n)}{\log y_4}\right), \tag{1.3}$$

where P's are polynomials. Recall that $\mu(n)$ denotes the Möbius function, also $\mu_2(n)$ and $\mu_3(n)$ will denote the coefficients in the Dirichlet series of $1/\zeta^2(s)$ and $1/\zeta^3(s)$, respectively, for $\mathrm{Re}(s) > 1$. Also, let $d_k(n)$ denote the number of ways an integer n can be written as a product of $k \geq 2$ fixed factors. Note that $d_1(n) = 1$ and that $d_2(n) = d(n)$ is the number of divisors of n. With this in mind, we define

$$\psi(s) := \psi_1(s) + \psi_2(s) + \psi_3(s) + \psi_4(s), \tag{1.4}$$

where

$$\psi_1(s) = \sum_{n \leq y_1} \frac{\mu(n)n^{\sigma_0 - 1/2}}{n^s} P_1[n] \tag{1.5}$$

introduced in [5],

$$\psi_2(s) = \chi(s + \tfrac{1}{2} - \sigma_0) \sum_{hk \leq y_2} \frac{\mu_2(h)h^{\sigma_0-1/2}k^{1/2-\sigma_0}}{h^s k^{1-s}} P_2[hk] \tag{1.6}$$

introduced in [2],

$$\psi_3(s) = \chi^2(s + \tfrac{1}{2} - \sigma_0) \sum_{hk \leq y_3} \frac{\mu_3(h)d(k)h^{\sigma_0-1/2}k^{1/2-\sigma_0}}{h^s k^{1-s}} P_3[hk] \tag{1.7}$$

introduced in the present paper, and