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Derivative operator and summation formulae involving generalized harmonic numbers

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ABSTRACT

In terms of the derivative operator and Baily's ${}_2F_1(\frac{1}{2})$-series identity, two kinds of summation formulae involving generalized harmonic numbers are established.

1. Introduction

For a complex variable x, define the shifted-factorial to be

$$(x)_0 = 0 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{when} \quad n \in \mathbb{N}.$$

Following Bailey [2], define the hypergeometric series to be

$${}_{1+r}F_s\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_s \end{matrix} \middle| z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k\cdots(a_r)_k}{(1)_k(b_1)_k\cdots(b_s)_k} z^k,$$

where $\{a_i\}_{i\geq 0}$ and $\{b_j\}_{j\geq 1}$ are complex parameters such that no zero factors appear in the denominators of the summand on the right hand side. Then Bailey's ${}_2F_1(\frac{1}{2})$-series identity (cf. [2, p. 11]) can be stated as

$${}_2F_1\left[\begin{matrix} a, 1-a \\ b \end{matrix} \middle| \frac{1}{2}\right] = \frac{\Gamma(\frac{b}{2})\Gamma(\frac{1+b}{2})}{\Gamma(\frac{b+a}{2})\Gamma(\frac{1+b-a}{2})}, \tag{1}$$

where $\Gamma(x)$ is the well-known gamma function

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$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad \text{with } Re(x) > 0.$$

For a complex variable x, define the generalized harmonic numbers to be

$$H_0(x) = 0 \quad \text{and} \quad H_n(x) = \sum_{k=1}^{n} \frac{1}{x+k} \quad \text{when} \quad n \in \mathbb{N}.$$

The case $x = 0$ of them are the classical harmonic numbers

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{when} \quad n \in \mathbb{N}.$$

For a differentiable function $f(x)$, define the derivative operator $\mathcal{D}_x$ by

$$\mathcal{D}_x f(x) = \frac{d}{dx} f(x).$$

In order to explain the relation of the derivative operator and generalized harmonic numbers, we introduce the following lemma.

Lemma 1. *Let x and $\{a_j, b_j, c_j, d_j\}_{j=1}^{k}$ be all complex numbers. Then*

$$\mathcal{D}_x \prod_{j=1}^{k} \frac{a_j x + b_j}{c_j x + d_j} = \prod_{j=1}^{k} \frac{a_j x + b_j}{c_j x + d_j} \sum_{j=1}^{k} \frac{a_j d_j - b_j c_j}{(a_j x + b_j)(c_j x + d_j)}.$$

Proof. It is not difficult to verify the case $k = 1$ of Lemma 1. Suppose that

$$\mathcal{D}_x \prod_{j=1}^{m} \frac{a_j x + b_j}{c_j x + d_j} = \prod_{j=1}^{m} \frac{a_j x + b_j}{c_j x + d_j} \sum_{j=1}^{m} \frac{a_j d_j - b_j c_j}{(a_j x + b_j)(c_j x + d_j)}$$

is true. We can proceed as follows:

$$\begin{aligned}
\mathcal{D}_x \prod_{j=1}^{m+1} \frac{a_j x + b_j}{c_j x + d_j} &= \mathcal{D}_x \left\{ \prod_{j=1}^{m} \frac{a_j x + b_j}{c_j x + d_j} \frac{a_{m+1} x + b_{m+1}}{c_{m+1} x + d_{m+1}} \right\} \\
&= \frac{a_{m+1} x + b_{m+1}}{c_{m+1} x + d_{m+1}} \mathcal{D}_x \prod_{j=1}^{m} \frac{a_j x + b_j}{c_j x + d_j} + \prod_{j=1}^{m} \frac{a_j x + b_j}{c_j x + d_j} \mathcal{D}_x \frac{a_{m+1} x + b_{m+1}}{c_{m+1} x + d_{m+1}} \\
&= \frac{a_{m+1} x + b_{m+1}}{c_{m+1} x + d_{m+1}} \prod_{j=1}^{m} \frac{a_j x + b_j}{c_j x + d_j} \sum_{j=1}^{m} \frac{a_j d_j - b_j c_j}{(a_j x + b_j)(c_j x + d_j)} \\
&\quad + \prod_{j=1}^{m} \frac{a_j x + b_j}{c_j x + d_j} \frac{a_{m+1} d_{m+1} - b_{m+1} c_{m+1}}{(c_{m+1} x + d_{m+1})^2} \\
&= \prod_{j=1}^{m+1} \frac{a_j x + b_j}{c_j x + d_j} \left\{ \sum_{j=1}^{m} \frac{a_j d_j - b_j c_j}{(a_j x + b_j)(c_j x + d_j)} + \frac{a_{m+1} d_{m+1} - b_{m+1} c_{m+1}}{(a_{m+1} x + b_{m+1})(c_{m+1} x + d_{m+1})} \right\} \\
&= \prod_{j=1}^{m+1} \frac{a_j x + b_j}{c_j x + d_j} \sum_{j=1}^{m+1} \frac{a_j d_j - b_j c_j}{(a_j x + b_j)(c_j x + d_j)}.
\end{aligned}$$

This proves Lemma 1 inductively. □