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The structure of optimal parameters for image restoration problems

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ABSTRACT

We study the qualitative properties of optimal regularisation parameters in variational models for image restoration. The parameters are solutions of bilevel optimisation problems with the image restoration problem as constraint. A general type of regulariser is considered, which encompasses total variation (TV), total generalised variation (TGV) and infimal-convolution total variation (ICTV). We prove that under certain conditions on the given data optimal parameters derived by bilevel optimisation problems exist. A crucial point in the existence proof turns out to be the boundedness of the optimal parameters away from 0 which we prove in this paper. The analysis is done on the original – in image restoration typically non-smooth variational problem – as well as on a smoothed approximation set in Hilbert space which is the one considered in numerical computations. For the smoothed bilevel problem we also prove that it Γ converges to the original problem as the smoothing vanishes. All analysis is done in function spaces rather than on the discretised learning problem.

1. Introduction

In this paper we consider the general variational image reconstruction problem that, given parameters $\alpha = (\alpha_1, \ldots, \alpha_N)$, $N \geq 1$, aims to compute an image

$$u_\alpha \in \arg\min_{u \in X} J(u; \alpha).$$

The image depends on α and belongs in our setting to a generic function space X. Here J is a generic energy modelling our prior knowledge on the image u_α. The quality of the solution u_α of variational imaging

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approaches like this one crucially relies on a good choice of the parameters α. We are particularly interested in the case

$$J(u;\alpha) = \Phi(Ku) + \sum_{j=1}^{N} \alpha_j \|A_j u\|_j,$$

with K a generic bounded forward operator, Φ a fidelity function, and A_j linear operators acting on u. The values $A_j u$ are penalised in the total variation or Radon norm $\|\mu\|_j = \|\mu\|_{\mathcal{M}(\Omega;\mathbb{R}^{m_j})}$, and combined constitute the image regulariser. In this context, α represents the regularisation parameter that balances the strength of regularisation against the fitness Φ of the solution to the idealised forward model K. The size of this parameter depends on the level of random noise and the properties of the forward operator. Choosing it too large results in over-regularisation of the solution and in turn may cause the loss of potentially important details in the image; choosing it too small under-regularises the solution and may result in a noisy and unstable output. In this work we will discuss and thoroughly analyse a bilevel optimisation approach that is able to determine the optimal choice of α in $J(;\alpha)$.

Recently, bilevel approaches for variational models have gained increasing attention in image processing and inverse problems in general. Based on prior knowledge of the problem in terms of a training set of image data and corresponding model solutions or knowledge of other model determinants such as the noise level, optimal reconstruction models are conceived by minimising a cost functional – called F in the sequel – constrained to the variational model in question. We will explain this approach in more detail in the next section. Before, let us give an account of the state of the art of bilevel optimisation for model learning. In machine learning bilevel optimisation is well established. It is a supervised learning method that optimally adapts itself to a given dataset of measurements and desirable solutions. In [41,42,28,29,18,19], for instance the authors consider bilevel optimisation for finite dimensional Markov random field (MRF) models. In inverse problems the optimal inversion and experimental acquisition setup is discussed in the context of optimal model design in works by Haber, Horesh and Tenorio [34,32,33], as well as Ghattas et al. [13,8]. Recently parameter learning in the context of functional variational regularisation models also entered the image processing community with works by the authors [23,14], Kunisch, Pock and co-workers [37,17] and Chung et al. [20]. A very interesting contribution can be found in a preprint by Fehrenbach et al. [31] where the authors determine an optimal regularisation procedure introducing particular knowledge of the noise distribution into the learning approach.

Apart from the work of the authors [23,14], all approaches for bilevel learning in image processing so far are formulated and optimised in the discrete setting. Our subsequent modelling, analysis and optimisation will be carried out in function space rather than on a discretisation of the variational model. In this context, a careful analysis of the bilevel problem is of great relevance for its application in image processing. The structure of optimal regularisers is important, among others, for the development of solution algorithms. In particular, if the parameters are bounded and lie in the interior of a closed connected set, then efficient optimisation methods can be used for solving the problem. Previous results on optimal parameters for inverse problems with partial differential equations have been obtained in, e.g., [16].

In this paper we study the qualitative structure of regularisation parameters arising as solutions from the bilevel optimisation of variational models. In our framework the variational models are typically convex but non-smooth and posed in Banach spaces. The total variation and total generalised variation regularisation models are particular instances. Alongside the optimisation of the non-smooth variational model, we also consider a smoothed approximation in Hilbert space which is typically the one considered in numerical computation. Under suitable conditions, we prove that – for both the original non-smooth optimisation problem as well as the regularised Hilbert space problem – the optimal regularisers are bounded and lie in the interior of the positive orthant. The conditions necessary to prove this turn out to be very natural