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A note on latticeability and algebrability

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ABSTRACT

Suppose A is a subset of a Banach lattice (Banach algebra) X. We look for "large" sublattices (resp. subalgebras) of A. If X is a Banach lattice, we prove: (1) If Y is a closed subspace of X of codimension at least n, then $(X\backslash Y) \cup \{0\}$ contains a sublattice of dimension n. (2) If Y is a closed infinite codimensional ideal in X, then $(X\backslash Y) \cup \{0\}$ contains a closed infinite dimensional sublattice. (3) If the order in X is induced by a 1-unconditional basis, and Y is a closed infinite codimensional subspace of X, then $(X\backslash Y) \cup \{0\}$ contains a closed infinite dimensional ideal. Further, we show that (4) $(\ell_p\backslash(\cup_{q<p}\ell_q)) \cup \{0\}$ contains a sublattice which is dense in ℓ_p, and that (5) the sets $L_1(\mathbb{T})\backslash(\cup_{p>1}L_p(\mathbb{T})) \cup \{0\}$ and $\mathcal{S}_\infty\backslash(\cup_{p<\infty}\mathcal{S}_p) \cup \{0\}$ contain a dense subalgebra with a continuum of free generators (here $\mathcal{S}_p$ denotes the Schatten p-space).

1. Introduction

We are motivated by the recent survey of lineability and spaceability [9]. Recall that $A \subset X$ is called *lineable* (*spaceable*, *densely lineable*) if $A \cup \{0\}$ contains an infinite dimensional subspace (resp. an infinite dimensional closed subspace, an infinite dimensional subspace dense in X).

If X is a Banach algebra, we say that $A \subset X$ is *algebrable* if $A \cup \{0\}$ contains a subalgebra B so that any family of generators of B is infinite. We say that A is *densely algebrable* if, in addition, B is dense in A.

In this paper, we search for large sublattices in subsets of Banach lattices. Suppose X is an infinite dimensional Banach lattice. A subset $A \subset X$ is (*completely*) *latticeable* if X contains a (complete) infinite dimensional sublattice Z so that $Z \subset A \cup \{0\}$.

As far as we know, the present paper is the first systematic investigation of latticeability. Even the term "latticeability" has not appeared previously – although, in fact, [1,21] produce atomic sublattices while proving the spaceability of certain sets in rearrangement invariant spaces. One should also mention the related notion of "coneability", studied in [14].

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In Section 2, we look for sublattices in complements of closed subspaces of a Banach lattice. We start in the finite codimensional case: if Y is a subspace of a Banach lattice X of codimension $n < \infty$, then $(X\backslash Y) \cup \{0\}$ contains an n-dimensional lattice (Theorem 2.1). Further, we show that complements of finite dimensional subspaces of a Banach lattice, and of infinite codimensional ideals, are completely latticeable (Propositions 2.4 and 2.8).

If the order of X is determined by a 1-unconditional basis, then Proposition 2.9 shows that $(X\backslash Y) \cup \{0\}$ contains a closed infinite dimensional ideal. Some partial results on complements of subspaces in Köthe function spaces are also established (Propositions 2.13, 2.15).

In Section 3 (Theorem 3.1) we prove that $\ell_p\backslash(\cup_{q<p}\ell_q)$ is *densely latticeable* – that is, $(\ell_p\backslash(\cup_{q<p}\ell_q)) \cup \{0\}$ contains a sublattice which is dense in ℓ_p.

In Section 4, we consider algebrability of subsets of a Banach algebra X. We prove that $L_1(\mathbb{T})\backslash(\cup_{p>1}L_p(\mathbb{T}))$ is densely maximally algebrable (Proposition 4.1) – that is, it contains a dense subalgebra W so that every set generating W must have the cardinality of continuum. Previously, similar results were obtained for $c_0\backslash(\cup_{q<\infty}\ell_q)$ [5] and $C_0(\mathbb{R})\backslash(\cup_{q<\infty}L_q(\mathbb{R}))$ [13]. In the non-commutative setting, we establish in Proposition 4.2 that $\mathcal{S}_\infty\backslash\big(\cup_{p<\infty}\mathcal{S}_p\big)$ is densely maximally algebrable (here $\mathcal{S}_p$ is a p-Schatten space on ℓ_2). Here, we should also note a plethora of results on spaceability and dense lineability of $L_p(\mu)\backslash(\cup_{q\in S}L_q(\mu))$ (where $S = (p, \infty)$, $(0, p)$, or $\mathbb{R}\backslash\{p\}$, and μ is not necessarily σ-finite), recently established in e.g. [6–8,11].

Finally, in Section 5, we prove (Proposition 5.1) that, for $1 < p < \infty$, the set of non-regular compact operators on ℓ_p is densely maximally lineable, and spaceable.

For the sake of simplicity, we work with real Banach lattices, while all Banach algebras are assumed to be complex. We use standard notation and results (see e.g. [3,19]). We denote by $\mathbf{B}(\cdot)$ the closed unit ball of a space. The unit circle is denoted by $\mathbb{T}$, and equipped with the translation-invariant Lebesgue probability measure μ_0.

2. Latticeability: complements of closed subspaces

We investigate the "largest" sublattice of X contained in the complement of a closed subspace $Z \subset X$. Note that, if Y is a closed subspace of a Banach space X of finite codimension, then clearly $(X\backslash Y) \cup \{0\}$ contains a subspace Z, with $\dim W = \dim X/Y$. Furthermore, if Y is a closed subspace of X with $\dim X/Y = \infty$, then $(X\backslash Y) \cup \{0\}$ contains a closed infinite dimensional subspace W. This result (attributed to N. Kalton) goes back to [25]. Related results concerning operator ranges were established in [18].

2.1. The finite codimensional case

In this subsection, we are assuming that Y is a finite codimensional subspace of a Banach lattice X. We are looking for a sublattice $W \subset X$, so that $W \cap Y = \{0\}$. We recall that any finite dimensional Banach lattice is spanned by its atoms, see [22, Section II.3] or [23, Proposition I.4.19].

Theorem 2.1. *If Y is a closed subspace of a Banach lattice X with $\dim X/Y \geq n$, then there exists an n-dimensional sublattice $W \subset X$ so that $W \cap Y = \{0\}$.*

Let us start with the case of finite dimensional X.

Lemma 2.2. *Suppose X is a Banach lattice of dimension n, and Y is a subspace of X of dimension $m < n$. Then X contains a sublattice W so that $\dim W = n - m$, and $W \cap Y = \{0\}$.*

Proof. As noted above, X is spanned by its atoms, so we can identify X (as a vector lattice) with $\mathbb{R}^n$ (or $\mathbb{C}^n$). By standard linear algebra, we can assume (up to relabeling) that $Y^\perp \subset \mathbb{R}^n$ has a basis $z_1 =$