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An estimate of the oscillation of harmonic reproducing kernels with applications

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ABSTRACT

We estimate the oscillation of harmonic reproducing kernels. As an application of this estimation we obtain a double integral characterization of harmonic Besov spaces.

1. Introduction

For $n \geq 2$, let $\mathbb{B}$ be the open unit ball in $\mathbb{R}^n$ and ν be the volume measure on $\mathbb{B}$ normalized so that $\nu(\mathbb{B}) = 1$. For $\alpha \in \mathbb{R}$, we define the weighted volume measures

$$d\nu_\alpha(x) = \frac{1}{V_\alpha}(1 - |x|^2)^\alpha d\nu(x).$$

These measures are finite only when $\alpha > -1$ and in this case we choose V_α so that $\nu_\alpha(\mathbb{B}) = 1$. For $\alpha \leq -1$, we set $V_\alpha = 1$. We denote the Lebesgue classes with respect to ν_α by L^p_α.

Let $h(\mathbb{B})$ be the space of all complex valued harmonic functions on $\mathbb{B}$. For $0 < p < \infty$ and $\alpha > -1$, the well-known harmonic weighted Bergman space b^p_α is $h(\mathbb{B}) \cap L^p_\alpha$. For $1 \leq p < \infty$, these spaces are extended to all $\alpha \in \mathbb{R}$ in [8], where they are called harmonic Besov spaces.

For $1 \leq p < \infty$ and $\alpha \in \mathbb{R}$, pick a nonnegative integer N so that

$$\alpha + pN > -1. \tag{1}$$

The harmonic Besov space b^p_α is the space of all $f \in h(\mathbb{B})$ such that

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$$(1-|x|^2)^N \partial^m f \in L^p_\alpha,$$

for every multi-index $m=(m_1,\cdots,m_n)$ with $|m|=N$. Here $|m|=m_1+\ldots+m_n$ and

$$\partial^m = \frac{\partial^{|m|}}{\partial x_1^{m_1}\ldots\partial x_n^{m_n}}.$$

The space b^p_α is independent of the choice of N as long as (1) is satisfied, and instead of partial derivatives one can also use radial derivatives or certain radial differential operators. These are studied in detail in [8].

When $\alpha>-1$, one can take $N=0$ and the resulting space is the usual harmonic weighted Bergman space. When $\alpha=-n$, the space b^p_{-n} is the standard harmonic Besov space. If also $p=2$, then b^2_{-n} is the harmonic Dirichlet space. If $\alpha=-1$ and $p=2$, then b^2_{-1} is the harmonic Hardy space h^2. We note that holomorphic analogues of the above spaces are studied in [11] and [21].

The spaces b^2_α, $\alpha\in\mathbb{R}$, are reproducing kernel Hilbert spaces with kernel $R_\alpha(x,y)$. These kernels are well-known for $\alpha>-1$ [14] and have been extended to all $\alpha\in\mathbb{R}$ in [7] and [8]. The main result of this work is the following estimate of the oscillation of reproducing kernels. We write

$$[x,y] := \sqrt{1-2x\cdot y+|x|^2|y|^2},$$

where $x\cdot y$ is the usual inner product of x and y in $\mathbb{R}^n$.

Theorem 1.1. *Let $\alpha>-n$ and $0\le\tau\le 1$. Then*

$$\frac{|R_\alpha(x,u)-R_\alpha(y,u)|}{|x-y|} \lesssim \frac{1}{[x,y]^{1-\tau}}\left(\frac{1}{[x,u]^{n+\alpha+\tau}}+\frac{1}{[y,u]^{n+\alpha+\tau}}\right),$$

for every $x,y,u\in\mathbb{B}$ with $x\neq y$.

We note that in Theorem 1.1 we can choose any τ between 0 and 1. This flexibility will later be very useful.

As an application of the above theorem we consider double integral characterizations of harmonic Besov spaces and extend previous results of [5]. For $f\in h(\mathbb{B})$, we define

$$Lf(x,y) := \frac{f(x)-f(y)}{|x-y|}, \quad x\neq y$$

and

$$\Lambda f(x,y) := \frac{f(x)-f(y)}{[x,y]}, \quad x,y\in\mathbb{B}.$$

Characterizations of holomorphic or harmonic Bergman, Besov or Bloch spaces in terms of Lf or Λf start with [9] (in the holomorphic case in Λf instead of $[x,y]$ one uses $|1-\langle x,y\rangle|$ with $\langle x,y\rangle$ being the inner product in $\mathbb{C}^n$). Further results with different ranges of α, p or n are obtained in [2,5,12,13,15–19].

The following two theorems are proved in [5].

Theorem A. *(See [5].) Suppose $\alpha>-1$, $0<p<n+\alpha$ and $f\in h(\mathbb{B})$. The following are equivalent:*

(a) $f\in b^p_\alpha$,
(b) $Lf\in L^p(\nu_\alpha\times\nu_\alpha)$,
(c) $\Lambda f\in L^p(\nu_\alpha\times\nu_\alpha)$.