J. Math. Anal. Appl. 434 (2016) 617–632

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Qualitative properties of traveling waves for nonlinear cellular neural networks with distributed delays

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ARTICLE INFO

Article history:
Received 24 March 2015
Available online 11 September 2015
Submitted by J. Shi

Keywords:
Nonlinear cellular neural network
Distributed delay
Traveling wave
Qualitative property

ABSTRACT

The purpose of this article is to study various qualitative properties of traveling waves for nonlinear cellular neural networks with distributed delays. The existence of monotone traveling waves of the model has been obtained in [25]. In this paper, we first determine the exact asymptotic behavior of the traveling waves at infinity. Then, the non-existence of the traveling waves is established. Finally, based on a strong comparison theorem, we prove the monotonicity and uniqueness (up to translation) of the traveling waves.

1. Introduction

In this article, we consider the traveling waves of the nonlinear cellular neural networks with distributed time delays (DCNN for short):

$$x_n'(t) = -x_n(t) + \sum_{i=1}^{m} \alpha_i \int_0^{\tau} J_i(s) f(x_{n-i}(t-s))ds$$
$$+ a \int_0^{\tau} J_{m+1}(s) f(x_n(t-s))ds + \sum_{j=1}^{\ell} \beta_j f(x_{n+j}(t)), \tag{1.1}$$

where $n \in \mathbb{Z}$, $t \in \mathbb{R}$, $m, \ell \in \mathbb{N}$, $\tau \geq 0$, $a \geq 0$, $\alpha_i \geq 0$, $i = 1, \cdots, m$ and $\beta_j \geq 0$, $j = 1, \cdots, \ell$ are given constants with $\alpha := \sum_{i=1}^{m} \alpha_i > 0$, $\beta := \sum_{j=1}^{\ell} \beta_j > 0$. It is assumed that the dynamics of each given cell depends on itself and m left or ℓ right neighbor cells where delay exists in self-feedback and left neighborhood interactions (see [3–5,25] for more details). In this article, we always assume that the output function f and the density function J_i, $i = 1, \cdots, m+1$ satisfy the following conditions:

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[1] Supported by the NSF of China (No. 11301407).

http://dx.doi.org/10.1016/j.jmaa.2015.09.021

(A$_1$) $f \in C([0,\infty),[0,\infty))$, $f \in C^1([K-\iota, K+\iota],[0,\infty))$ for some $\iota \in (0,K)$, $f(0)=0$, $f''(0)$ exists,

$$(a+\alpha+\beta)f(K)=K \quad \text{and} \quad (a+\alpha+\beta)f(u)>u \text{ for } u\in(0,K),$$

where $K>0$ is a constant;

(A$_2$) $f(u) \le f'(0)u$ for $u \in [0,K]$, $(a+\beta)f'(0)>1$, $(a+\alpha+\beta)f'(K)<1$ and $f(u)$ is non-decreasing and Lipschitz continuous on $[0,K]$;

(A$_3$) $J_i \in L^1([0,\tau])$ is a nonnegative function satisfying $\int_0^\tau J_i(s)ds = 1$, $i=1,\cdots,m+1$.

In the past decades, the traveling waves for various versions of (delayed) cellular neural network systems with *piecewise-linear* output functions have been extensively studied due to their significant nature, see e.g., [10–13,15,16,22]. Recently, Liu et al. [17] considered the existence of monotone traveling waves for a class of delayed cellular neural networks model with *nonlinear* output functions. Yu et al. [25] further extended the existence results of monotone traveling waves in [10,12,15–17,22] to a more general DCNN model which includes (1.1) as a special case. On the other hand, to formulate and understand the natural phenomena through traveling waves, it is very important to investigate their other qualitative properties, such as asymptotic behavior, monotonicity and uniqueness. However, to the best of our knowledge, there have been no results on such qualitative properties of the traveling waves for the DCNN models. Resolving these issues represents a main contribution of the current study.

More precisely, in this paper, we shall consider the asymptotic behavior, non-existence, monotonicity and uniqueness of traveling waves of (1.1). It is well-known that a traveling wave solution of (1.1) refers to a special translation invariant solution with the form $x_n(t)=\phi(n+ct)$, $n\in\mathbb{Z}$, $t\in\mathbb{R}$ for a wave profile $\phi(\cdot):\mathbb{R}\to[0,K]$ with an unknown wave speed $c\in\mathbb{R}$. If $\phi(\cdot)$ is monotone, then we say ϕ is a *traveling (wave) front*. Letting $\xi=n+ct$, it is easy to see that such a profile must satisfy the following equation

$$\begin{aligned} c\phi'(\xi) &= -\phi(\xi) + \sum_{i=1}^{m}\alpha_i \int_0^\tau J_i(s) f(\phi(\xi-i-cs))ds \\ &\quad + a\int_0^\tau J_{m+1}(s) f(\phi(\xi-cs))ds + \sum_{j=1}^{\ell}\beta_j f(\phi(\xi+j)). \end{aligned} \tag{1.2}$$

In order to state our results later, we first recall the existence results on traveling waves of (1.1), see Yu et al. [25, Theorem 1.1].

Proposition 1.1 *(Existence). Assume* (A$_1$)–(A$_3$). *Then there exists a $c_*>0$ such that for each $c\ge c_*$, equation* (1.1) *has a leftward monotone traveling wave $\phi_c(n+ct)$ which satisfies*

$$\phi_c(-\infty)=0 \text{ and } \phi_c(+\infty)=K. \tag{1.3}$$

For a traveling wave, its asymptotic behavior near $\pm\infty$ determines its important properties of the wave. According to the existence result of Proposition 1.1, we apply the Ikehara's theorem (cf. Proposition 2.3) to investigate the rate of approach to the steady states as $\xi\to\pm\infty$ of a *generic* traveling wave. This method has been adopted by many researchers for various evolution equations, see e.g., [1,6,8,14,20,21,24,26]. For simplicity, a traveling wave of (1.1) always refers to a solution of (1.2) satisfying (1.3).

Theorem 1.2 *(Asymptotic behavior). Assume that* (A$_1$)–(A$_3$) *hold and $f(u)$ is strictly increasing on $[0,K]$. Let (ϕ,c) be a traveling wave of* (1.1) *with $c\ge c_*$. Then the following statements hold:*