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The center problem. A view from the normal form theory

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A B S T R A C T

In this work, we analyze some aspects of the center problem from the perspective of the normal form theory. We provide alternative proofs of some well known results in the case of non-vanishing linear part (nondegenerate and nilpotent centers). Moreover, some new results are also derived. In particular, we show that the unique characterization for centers in this situation is the orbital reversibility. Also, we include some examples illustrating that this characterization is no longer valid for the case of null linear part.

1. Introduction

The center problem for planar analytic autonomous systems with an equilibrium point located at the origin

$$\begin{aligned} \dot{x} &= F(x, y), \\ \dot{y} &= G(x, y), \end{aligned} \tag{1.1}$$

where $F(0, 0) = 0$, $G(0, 0) = 0$, constitutes a classical question in the field of ordinary differential equations. It can be formulated as follows:

> Determine conditions under which all trajectories of the above system in a certain punctured neighborhood of the equilibrium at the origin are closed curves.

This problem was posed by Poincaré, and it has been considered in a great number of works (see, e.g., [6,13–15,20,24–26,32,34,37,39–41]) and recent monographs (for instance, see [16,36]). Although the center

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problem has been solved for important subclasses of planar systems, it remains open in its full generality. A previous issue to be considered in the center problem is to establish the monodromy of the equilibrium point, that consists of determining if there is no orbit tending to the equilibrium point with definite tangent at this point (see [11]). The monodromy problem was solved in the nilpotent case in [9,10] and in the general case in [5,18,30]. In the analytic case, if the equilibrium is monodromic, then it is either a center or a focus, and the center problem (stability problem) consists of distinguishing between both configurations (see [17,23]).

Regarding the linear part of system (1.1), the centers can be classified into nondegenerate (also called elementary), if the eigenvalues are different from zero (in fact, they are a pair of pure imaginary eigenvalues), or degenerate (nonelementary) when both eigenvalues are zero. In this last case, they are called nilpotent when the linearization matrix is non-zero and null centers otherwise. In other words, if system (1.1) has a center at the origin then, after a linear transformation and a time-rescaling, its linear part can be reduced to one of the following cases: $(-y, x)^T$, $(y, 0)^T$ or $(0, 0)^T$.

Our contribution to the center problem is based on results from the orbital-normal form theory. We provide alternative proofs of some known geometric criteria characterizing nondegenerate centers (for instance, the Lyapunov–Poincaré Center Theorem) and degenerate nilpotent centers. In particular, we characterize the nondegenerate centers in terms of the integrability, the reversibility, the existence of inverse integrating factors or the existence of Lie symmetries. We show that none of these conditions is valid for nilpotent centers, that are characterized by the orbital reversibility. Hence, the unique characterization that share the nondegenerate or nilpotent centers is the orbital reversibility. Moreover, we include some examples revealing that this characterization is no longer valid for null centers, so that in this case there are no criteria for centers related to the concepts of reversibility, integrability, inverse integrating factors or Lie symmetries. Anyway, some new cases of some degenerate centers are derived.

This paper is organized as follows. Next section introduces some notations and results needed later, mainly about orbital normal forms for planar systems using quasi-homogeneous expansions of the vector field. Next, in Section 3 we derive some well-known results for nondegenerate centers, where we provide alternative proofs based on orbital normal forms. Later, we consider nilpotent systems in Section 4. The use of an adequate orbital normal form provides results characterizing the monodromy, the integrability and the centers in this situation. Finally, in Section 5 we deal with a particular case of null degenerate centers which serves as a counterexample for the characterizations of the above cases.

2. Background, notations and basic definitions

The key in the orbital normal form theory is to expand the vector field (in the classical theory, a Taylor expansion is used) and simplify it degree by degree through near-identity transformations and time-reparametrizations. This procedure allows to obtain a simple representative of the equivalence class of systems, with the equivalence relation defined by coordinate transformations and time-reparametrizations. Our idea is to use this orbitally equivalent representative to obtain results on the center problem.

Let us summarize some notations and results, which can be found in detail in [3], that we use here in order to determine adequate orbital normal forms for nondegenerate and degenerate centers.

Let us write the system (1.1) compactly in vector notation as

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad \text{being } \mathbf{x} = (x, y)^T \in \mathbb{R}^2, \ \mathbf{F} = (F, G)^T.$$

Let us consider a type $\mathbf{t} = (t_1, t_2) \in \mathbb{N}^2$. A scalar polynomial $p : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is quasi-homogeneous of type $\mathbf{t}$ and degree k if $p(\varepsilon^{t_1} x, \varepsilon^{t_2} y) = \varepsilon^k p(x, y)$ for all $\varepsilon \in \mathbb{R}$. The set of scalar quasi-homogeneous polynomials of type $\mathbf{t}$ and degree k is denoted by $\mathcal{P}_k^{\mathbf{t}}$. The vector field $\mathbf{F}$ is called quasi-homogeneous of type $\mathbf{t}$ and degree k