J. Math. Anal. Appl. 434 (2016) 698–712

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Differentiable positive definite functions on two-point homogeneous spaces

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A R T I C L E I N F O

Article history:
Received 7 January 2015
Available online 25 September 2015
Submitted by K. Driver

Keywords:
Positive definite kernels
Isotropic kernels
Homogeneous spaces
Jacobi polynomials
Differentiability

A B S T R A C T

In this paper we study continuous kernels on compact two point homogeneous spaces which are positive definite and zonal (isotropic). Such kernels were characterized by R. Gangolli some forty years ago and are very useful for solving scattered data interpolation problems on the spaces. In the case the space is the d-dimensional unit sphere, J. Ziegel showed in 2013 that the radial part of a continuous positive definite and zonal kernel is continuously differentiable up to order $\lfloor (d-1)/2 \rfloor$ in the interior of its domain. The main issue here is to obtain a similar result for all the other compact two point homogeneous spaces.

1. Introduction

Let $\mathbb{M}^d$ denote a d dimensional compact two-point homogeneous space. It is well known that spaces of this type belong to one of the following categories [5]: the unit spheres S^d, $d = 1, 2, \ldots$, the real projective spaces $\mathbb{P}^d(\mathbb{R})$, $d = 2, 3, \ldots$, the complex projective spaces $\mathbb{P}^d(\mathbb{C})$, $d = 4, 6, \ldots$, the quaternionic projective spaces $\mathbb{P}^d(\mathbb{H})$, $d = 8, 12, \ldots$, and the Cayley projective plane $\mathbb{P}^d(Cay)$, $d = 16$. Standard references containing all the basics about two-point homogeneous spaces that will be needed here are [6,9] and others mentioned there.

In this paper, we will deal with real, continuous, positive definite and zonal (isotropic) kernels on $\mathbb{M}^d$. The positive definiteness of a kernel K on $\mathbb{M}^d$ will be the standard one: it requires that

$$\sum_{\mu,\nu=1}^{n} c_\mu c_\nu K(x_\mu, x_\nu) \geq 0,$$

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http://dx.doi.org/10.1016/j.jmaa.2015.09.040

whenever n is a positive integer, $x_1, x_2, \ldots, x_n$ are distinct points on $\mathbb{M}^d$ and $c_1, c_2, \ldots, c_n$ are real scalars. The continuity of K can be defined through the usual (geodesic) distance on $\mathbb{M}^d$, here denoted by $|xy|$, $x, y \in \mathbb{M}^d$. We will assume such distance is normalized so that all geodesics on $\mathbb{M}^d$ have the same length 2π. Since $\mathbb{M}^d$ possesses a group of motions G_d which takes any pair of points (x,y) to (z,w) when $|xy| = |zw|$, zonality of a kernel K on $\mathbb{M}^d$ will refer to the property

$$K(x,y) = K(Ax, Ay), \quad x, y \in \mathbb{M}^d, \quad A \in G_d.$$

A zonal kernel K on $\mathbb{M}^d$ can be written in the form

$$K(x,y) = K_r^d(\cos|xy|/2), \quad x, y \in \mathbb{M}^d,$$

for some function $K_r^d : [-1,1] \to \mathbb{R}$, the *radial* or *isotropic part* of K. A result due to Gangolli [2] established that a continuous zonal kernel K on $\mathbb{M}^d$ is positive definite if and only if

$$K_r^d(t) = \sum_{k=0}^{\infty} a_k^{(d-2)/2,\beta} P_k^{(d-2)/2,\beta}(t), \quad t \in [-1,1], \tag{1}$$

in which $a_k^{(d-2)/2,\beta} \in [0,\infty)$, $k \in \mathbb{Z}_+$ and $\sum_{k=0}^{\infty} a_k^{(d-2)/2,\beta} P_k^{(d-2)/2,\beta}(1) < \infty$. Here, $\beta = (d-2)/2$, $-1/2, 0, 1, 3$, depending on the respective category $\mathbb{M}^d$ belongs to, among the five we have mentioned in the beginning of the paper. The symbol $P_k^{(d-2)/2,\beta}$ stands for the Jacobi polynomial of degree k associated with the pair $((d-2)/2, \beta)$.

Gneiting [4] conjectured that the radial part of a continuous, positive definite and zonal kernel on S^d is continuously differentiable in $(-1,1)$ up to order $\lfloor (d-1)/2 \rfloor$ (largest integer not greater than $(d-1)/2$). The conjecture was ratified by Ziegel [11] who also proved that the differentiability order in Gneiting's conjecture is best possible in the case d odd. In other words, she proved that if d is odd, there exists a continuous, positive definite and zonal kernel K on S^d for which the $\lfloor (d-1)/2 \rfloor$ derivative of K_r^d is not continuously differentiable. In addition, she analyzed some specific examples to show that the one side derivatives of K_r^d at the extreme points -1 or 1 can either take finite values or be infinite.

Menegatto [7] added to Ziegel's results establishing similar results in the complex setting, that is, replacing the unit sphere S^d with the unit sphere in $\mathbb{C}^d$ and allowing the positive definite functions to assume complex values. The radial part of a positive definite kernel on complex spheres depend upon a complex variable z and its conjugate $\bar{z}$. As so, in the complex setting, derivatives can be considered with respect to these two variables. The deduction of the results in this complex version demanded quite a number of changes in the procedure used in S^d, some of them not obvious. While Ziegel's arguments were based upon recurrence formulas for Fourier–Gegenbauer coefficients of certain continuous functions on $[-1,1]$, Menegatto's invoked some similar properties for the double indexed coefficients of a continuous function on the unit disk with respect to disk (Zernike) polynomials.

This is the point where we state the main result to be proved in this paper, a first step extension of the results described above to compact two point homogeneous spaces.

Theorem 1.1. *If K is a continuous, positive definite and zonal kernel on $\mathbb{M}^d$, then the radial part K_r^d of K is continuously differentiable on $(-1,1)$. The derivative $(K_r^d)'$ of K_r^d in $(-1,1)$ satisfies a relation of the form*

$$(1-t^2)(K_r^d)'(t) = f_1(t) - f_2(t), \quad t \in (-1,1),$$

in which f_1 and f_2 are the radial parts of two continuous, positive definite and zonal kernels on some compact two-point homogeneous space $\mathbb{M}$ which is isometrically embedded in $\mathbb{M}^d$. The specifics on d and $\mathbb{M}$ in each case are these ones: