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Total nonnegativity of matrices related to polynomial roots and poles of rational functions

Mohammad Adm [a,1], Jürgen Garloff [b,a,*], Jihad Titi [a]

[a] Department of Mathematics and Statistics, University of Konstanz, D-78464 Konstanz, Germany
[b] Institute for Applied Research, University of Applied Sciences/HTWG Konstanz, D-78405 Konstanz, Germany

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ABSTRACT

In this paper totally nonnegative (positive) matrices are considered which are matrices having all their minors nonnegative (positive); the almost totally positive matrices form a class between the totally nonnegative matrices and the totally positive ones. An efficient determinantal test based on the Cauchon algorithm for checking a given matrix for falling in one of these three classes of matrices is applied to matrices which are related to roots of polynomials and poles of rational functions, specifically the Hankel matrix associated with the Laurent series at infinity of a rational function and matrices of Hurwitz type associated with polynomials. In both cases it is concluded from properties of one or two finite sections of the infinite matrix that the infinite matrix itself has these or related properties. Then the results are applied to derive a sufficient condition for the Hurwitz stability of an interval family of polynomials. Finally, interval problems for a subclass of the rational functions, viz. *R*-functions, are investigated. These problems include invariance of exclusively positive poles and exclusively negative roots in the presence of variation of the coefficients of the polynomials within given intervals.

1. Introduction

In this paper we consider matrices which are related to stability of polynomials and to the localization of the poles and zeros of rational functions. Specifically, in the case of polynomials we focus on matrices of Hurwitz type which are closely related to (Hurwitz) stability of a polynomial, i.e., to the property that all zeros are contained in the open left half of the complex plane. In the case of rational functions we focus on R-functions of negative type, i.e., functions which map the open upper half-plane of the complex plane to the open lower half-plane. For references and properties of this important class of functions the reader is referred to the survey given in [14]. In the polynomial as well as in the rational case we are interested in

* Corresponding author.
E-mail addresses: mjamathe@yahoo.com (M. Adm), garloff@htwg-konstanz.de (J. Garloff), jihadtiti@yahoo.com (J. Titi).

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interval problems which arise when the polynomial coefficients are due to uncertainty caused by, e.g., data uncertainties but can be bounded in intervals. For background material from control theory and practical applications see [4,5]. It turns out that certain properties concerning the zeros and the poles remain in force through all the coefficient intervals if up to four polynomials of the entire family have certain properties. Typically, the coefficients of these polynomials alternate in attaining the endpoints of the coefficient intervals. This up-and-down behavior corresponds to a checkerboard pattern of the entries of the associated matrices.

The underlying property of all the matrices considered in this paper is that all their minors are nonnegative. Such matrices are called *totally nonnegative*. For properties of these matrices the reader is referred to the monographs [6,18]. In [2] we derive an efficient determinantal test based on the Cauchon algorithm [12,17] for checking a given matrix for total nonnegativity and related properties. In this paper we apply this test to the matrices mentioned above. To solve the related interval problems we make use of a result in [1] by which from the nonsingularity and the total nonnegativity of two matrices we can infer that all matrices lying between these two matrices are nonsingular and totally nonnegative, too. Here 'between' is meant in the sense of the checkerboard ordering, see above.

The organization of our paper is as follows. In the next section we first introduce the notation and the definitions and recall then some properties of the totally nonnegative matrices which we will use in our paper. We also briefly recall the Cauchon algorithm and characterizations of two subclasses of the totally nonnegative matrices. In Section 3 we show that from properties of finite sections of an infinite Hankel matrix or matrix of Hurwitz type we may conclude that the infinite matrix itself possesses these or related properties. We also derive a sufficient condition for the stability of an interval family of polynomials. In Section 4 we present interval problems related to R-functions.

2. Notation and auxiliary results

For nonnegative integers k, n we denote by $Q_{k,n}$ the set of all strictly increasing sequences of k integers chosen from $\{1, 2, \ldots, n\}$. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in Q_{k,n}$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_l) \in Q_{l,m}$, we denote by $A[\alpha|\beta]$ the $k \times l$ submatrix of A contained in the rows indexed by $\alpha_1, \alpha_2, \ldots, \alpha_k$ and columns indexed by $\beta_1, \beta_2, \ldots, \beta_l$. We suppress the parentheses when we enumerate the indices implicitly. When $\alpha = \beta$, the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$; if $\alpha = \beta = (1, \ldots, k)$ the submatrix is called a *leading* principal submatrix. We denote by $\mid \alpha \mid$ the number of members of α. If $k = l$ and $A[\alpha|\beta]$ is formed from consecutive rows and columns of A then it is called *contiguous* and its determinant is termed a *contiguous* minor. A matrix $A \in \mathbb{R}^{m,n}$ is called *totally positive* (abbreviated *TP* henceforth) and *totally nonnegative* (abbreviated *TN*) if $\det A[\alpha|\beta] > 0$ and $\det A[\alpha|\beta] \geq 0$, respectively, for all $\alpha \in Q_{k,m}$, $\beta \in Q_{k,n}$. If $A \in \mathbb{R}^{n,n}$ is *TN* and in addition nonsingular we write A is *NsTN*. In [9] Gasca et al. define the following class of matrices intermediate between the totally nonnegative and the totally positive matrices. If $A \in \mathbb{R}^{m,n}$ is *TN* it is said to be *almost totally positive* (abbreviated *ATP*) if it satisfies the following two conditions:

(i) Any contiguous minor of A is positive if and only if the diagonal entries of the corresponding submatrix are positive.
(ii) In the case that A has a zero row or column, the subsequent rows or columns also are zero, respectively.

It was proven in [9] that if A is *ATP* then (i) holds for any minor of A (not only for the contiguous ones). If A is *ATP* and in addition it is nonsingular then we write A is *NsATP*. These matrices were also introduced independently in [11]. For further properties see [10].

We endow $\mathbb{R}^{m,n}$ with two partial orderings: Firstly, with the usual entry-wise ordering ($A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{m,n}$)

$$A \leq B \Leftrightarrow a_{ij} \leq b_{ij}, \;\; i = 1, \ldots, m, \; j = 1, \ldots, n.$$