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Local estimates for functionals depending on the gradient with a perturbation

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ABSTRACT

This paper concerns minimization problems from Calculus of Variations depending on the gradient and with a linear perturbation. Inspired in qualitative properties that are valid for elliptic partial differential equations, it presents some local estimates near non extremum points as well as extremum points. These estimates are inspired on a class of functions given by A. Cellina in [2]. Also, a comparison result with respect to these functions is presented. Finally, some local estimates are obtained for the difference between the supremum and the infimum of any solution to the problems considered.

0. Introduction

Many connections between solutions to Partial Differential equations and solutions to minimization problems in Calculus of Variations have been made, especially regarding its qualitative properties. The results on classical cases when we consider for instance Laplace and Poisson equations are good examples of how those connections started to be made. It is known that Laplace equation

$$\Delta u(x) = 0, x \in \Omega, \tag{1}$$

can be seen as the Euler–Lagrange equation (a necessary condition for minimizers) when we consider the problem of minimizing the functional

$$\int_\Omega \left(\frac{1}{2}\|\nabla u(x)\|^2\right) dx, \tag{2}$$

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and in fact they have the same solutions, called harmonic functions. The validity of some qualitative properties as the Strong Maximum Principle and Harnack Inequality were proved for harmonic function as we can see in [3]. They were proved for more general elliptic equations (see for instance the work developed by James Serrin and Patrizia Pucci in [8]). It makes sense to ask whether these properties were proved for more general minimization problems than those involving the functional (2).

Considering the more general problem of minimizing integrals of the form,

$$(P) \qquad \min\left\{ \int_\Omega f(\|\nabla u(x)\|)\mathrm{d}x : u(\cdot) \in u^0(\cdot) + W_0^{1,1}(\Omega) \right\},$$

the Strong Maximum Principle, which states that if any nonnegative solution $\bar{u}$ to (P) is equal to zero on some interior point of Ω, then $\bar{u} \equiv 0$ on Ω was proved by A. Cellina in [1]. More general versions of the Strong Maximum Principle were presented by V.V. Goncharov and myself in [4] and [5].

If we consider, for $l : \Omega \to \mathbb{R}$, $l \neq 0$ on Ω, the Poisson equation

$$\Delta u(x) = l(x), x \in \Omega, \tag{3}$$

which can be seen as the Euler–Lagrange equation for the problem of minimizing the functional

$$\int_\Omega \left(\frac{1}{2}\|\nabla u(x)\|^2 + l(x)u(x) \right) dx, \tag{4}$$

the Strong Maximum Principle (SMP) in the classical sense does not make sense anymore. In fact, if (SMP) was valid, for $\bar{u}$ solution to (3), $\exists \bar{x} \in \Omega : \bar{u} = 0 \Rightarrow \bar{u}(x) = 0\ \forall x \in \Omega$. Then, obviously $\Delta\bar{u}(x) = 0$ on Ω, and like this $l(x) = 0$ on Ω, which contradicts the formulation of the Poisson equation. In this case, D. Gilbarg and N. Trudinger presented in [3] (Theorem 3.7) some pointwise estimates for solutions to (3).

In variational context, considering minimization problems of the type

$$(\bar{P}_i) \qquad \min\left\{ \int_\Omega [f(\|\nabla u(x)\|) + l_i u(x)]\mathrm{d}x : u(\cdot) \in u^0(\cdot) + W_0^{1,1}(\Omega) \right\}$$

with $i \in \{1, 2\}$, $l_1 > 0$ and $l_2 < 0$, Arrigo Cellina presented in [2] classes of solutions to $(\bar{P}_i)$, $i \in \{1, 2\}$. In [7], V.V. Goncharov, A. Cellina and myself presented local estimates for solutions to the same problems with an interesting feature: local estimates for $(\bar{P}_1)$ are obtained through solutions to $(\bar{P}_2)$ and vice-versa.

In this paper we consider more general lagrangians where, instead of considering a constant $l_1 > 0$ ($l_2 < 0$), we consider a function

$$l_1 : \Omega \to \mathbb{R}^+ \quad (l_2 : \Omega \to \mathbb{R}^-)$$

such that $\inf_{x\in\Omega} l_1(x) > 0$ (respectively, $\sup_{x\in\Omega} l_2(x) < 0$). As in [7] we prove local estimates in neighborhoods of nonextremum points as well as in neighborhoods of extremum points, inspired by the family of solutions presented by A. Cellina in [2].

In the first section, we introduce our problems and several notions of convex analysis to be used throughout this paper. We will introduce the class of functions that, in second section will be used to present a comparison result, and in the third section, prove local estimates near some nonextremum points. In the fourth section some local estimates for extremum points through these concrete functions are presented. We finally prove that it is possible to obtain a local estimate for the difference between the supremum and infimum of any solution.