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Large solutions for an elliptic equation with a nonhomogeneous term

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A B S T R A C T

In this paper we study the elliptic boundary blow-up problem $\Delta u = |u|^{p-1}u + h(x)$ in Ω, where $p > 1$, Ω is a smooth bounded domain of $\mathbb{R}^N$ and $h \in C(\Omega)$. We are concerned with existence and uniqueness of solutions for quite general functions h, which may change sign and are not necessarily bounded near $\partial\Omega$.

1. Introduction and results

In this paper, we perform an analysis of the following semilinear problem with boundary blow-up:

$$\begin{cases} \Delta u = |u|^{p-1}u + h(x) & \text{in } \Omega \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $p > 1$ and $h \in C(\Omega)$. The case $h = 0$ is well understood and has been widely analyzed. By using for instance the results in [11] and [2], it is known that there exists a unique *positive* solution $U \in C^\infty(\Omega)$ such that

$$U(x) \sim (\alpha(\alpha+1))^{\frac{1}{p-1}} d(x)^{-\alpha} \quad \text{as } x \to \partial\Omega,$$

where $d(x) = \text{dist}(x, \partial\Omega)$ and

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$$\alpha = \frac{2}{p-1} \tag{1.2}$$

(also, it can be easily seen that no sign-changing solutions may exist). However, a complete analysis of problem (1.1) for general continuous functions h has not been carried out, as far as we know, except for some partial results which we now comment.

In [4] the existence and uniqueness of a positive solution was shown in the case $h \le 0$ and with a suitable growth near $\partial\Omega$, while in [14], the case where h is also negative but small when compared with $d(x)^{-\frac{2p}{p-1}}$ near $\partial\Omega$ was considered. Finally, in [15] some particular positive functions h vanishing on the boundary were analyzed, whereas in [8] the case $h(x) = -C_0 d(x)^{-\gamma}$, with $C_0, \gamma > 0$, was studied. But, as it stands, there does not seem to be available even an existence result dealing with changing sign functions h. Moreover, when h is singular and positive near $\partial\Omega$ nothing seems to be known even with regard to existence. We refer the interested reader to the papers [5] and [3] for related developments (see also the book [9] for more on large solutions).

Therefore our objective in the present paper is to partly fill this gap and try to understand a little better the features of problem (1.1). Let us remark first that, especially in the case where h is positive somewhere, it might not be reasonable to look for nonnegative solutions of (1.1). And even if a nonnegative solution exists, it is not necessarily positive, since the strong maximum principle can only be applied if $h \le 0$.

Let us also mention that we will always be dealing with weak solutions $u \in H^1_{\rm loc}(\Omega)$, which according to standard regularity verify $u \in C^{1,\eta}_{\rm loc}(\Omega)$ for every $\eta \in (0,1)$ (cf. [10]).

We begin by considering existence of solutions of (1.1). When the function h is negative in Ω, existence (of nonnegative solutions) holds in general, so that only the positive part of h needs to be restricted. Actually, it is only necessary that h does not grow too fast near $\partial\Omega$ in order to obtain a solution. To make this precise, consider the function

$$f(t) = \alpha(\alpha+1)t - t^p, \quad t > 0,$$

where α is given by (1.2). Since $p > 1$, it is clear that f is bounded from above, so we can denote $\Lambda = \sup_{t>0} f(t)$. Although its exact value will be of no importance for our proofs, we just mention that with the aid of standard calculus it can be seen that

$$\Lambda = (p-1)\left(\frac{\alpha(\alpha+1)}{p}\right)^{\frac{p}{p-1}}.$$

Let us state our result on existence of solutions.

Theorem 1. *Let Ω be a bounded C^2 domain of $\mathbb{R}^N$ and assume $p > 1$ and $h \in C(\Omega)$ verifies*

$$L := \limsup_{x \to \partial\Omega} d(x)^{\frac{2p}{p-1}} h(x) < \Lambda. \tag{1.3}$$

Then problem (1.1) admits at least a solution u. Moreover, u is the maximal solution of the problem and it verifies

$$\liminf_{x \to \partial\Omega} d(x)^{\frac{2}{p-1}} u(x) \ge \xi, \tag{1.4}$$

where ξ is the largest root of the equation $f(t) = L$.

Let us mention that the obtention of a minimal solution (1.1) does not seem to be an easy task in general. As a rule, such a solution is obtained as the limit of solutions of the same equation with finite boundary