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Integrability of complex planar systems with homogeneous nonlinearities

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A B S T R A C T

In this paper we obtain sufficient conditions for the existence of a local analytic first integral for a family of quintic systems having homogeneous nonlinearities. The family studied in this work is the largest one classified until now for systems with such nonlinearities. We propose also an approach to find reversible systems within polynomial families of Lotka–Volterra systems with homogeneous nonlinearities.

1. Introduction and statement of the main results

The integrability problem for systems of differential equations is one of the main problems in the qualitative theory of differential systems. In fact, integrability, although a rare phenomenon, is of great importance due to applications in the bifurcation theory. In the study of mathematical models it is important to detect rare systems that are integrable, since perturbations of such systems exhibit a rich behavior of bifurcations.

From the beginning of the last century many papers have been devoted to studies on the existence of a local analytic first integral in a neighborhood of a singular point for real autonomous polynomial differential systems in the plane, see for instance [2,4,22,26] and references therein. The most studied case is the singular point with pure imaginary eigenvalues of the matrix of linear approximation. Limiting our consideration to polynomial systems we can write such systems in the form

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$$\dot{u} = -v + \sum_{i+j=2}^{n} \alpha_{ij} u^i v^j, \qquad \dot{v} = u + \sum_{i+j=2}^{n} \beta_{ij} u^i v^j. \tag{1}$$

By the Poincaré–Lyapunov theorem (see e.g. [26]) the local integrability of system (1) is equivalent to the existence of a center at the origin. That means, system (1) admits a local analytic first integral if and only if all solutions are periodic in a neighborhood of the origin. To study local integrability of (1) it is convenient to enclose this family into the more general family of complex differential systems (see e.g. [26, §3.2] for more details):

$$\dot{x} = ix + \sum_{j+k=2}^{n} X_{jk}\, x^{j+1} y^k, \qquad \dot{y} = -iy + \sum_{j+k=2}^{n} Y_{jk}\, x^j y^{k+1}, \tag{2}$$

where X_{jk} and Y_{jk} are complex parameters. System (2) is equivalent to system (1) in the case that $x = \bar{y}$ and $X_{jk} = \bar{Y}_{kj}$ (if these conditions are satisfied we say that (2) is a complexification of (1)). After the change of time $t \mapsto it$ system (2) takes the form

$$\begin{aligned} \dot{x} &= \mathcal{X}(x,y) = x - \sum_{j+k=2}^{n} a_{jk}\, x^{j+1} y^k, \\ \dot{y} &= \mathcal{Y}(x,y) = -y + \sum_{j+k=2}^{n} b_{jk}\, x^j y^{k+1}. \end{aligned} \tag{3}$$

For system (3) one can always find a function of the form

$$\Psi(x,y) = xy + \sum_{j+k\geq 3} \psi_{j,k}\, x^j y^k, \tag{4}$$

such that

$$\dot{\Psi}(x,y) = \sum_{j\geq 1} g_{jj} (xy)^{j+1} = xy \sum_{j\geq 1} g_{jj} x^j y^j, \tag{5}$$

where $g_{jj} \in \mathbb{Q}[a,b]$ (here and below $\mathbb{Q}[a,b]$ stands for the ring of polynomials with rational coefficients in parameters a_{jk}, b_{jk} of system (3)). By the analogy with the real case we call the polynomial g_{kk} the *k-th focus quantity of system* (3). The ideal $\mathcal{B} = \langle g_{11}, g_{22}, g_{33}, \dots \rangle$ generated by focus quantities is called the *Bautin ideal* of system (3). Polynomials g_{kk} are not uniquely defined, but we are interested in determining the variety of $\mathcal{B}$,[1] which is the same for any choice of focus quantities g_{jj} satisfying (5) (see e.g. Theorem 3.3.5 of [26]). In the case when $\dot{\Psi} \equiv 0$ we say that the origin of system (3) is an *integrable complex saddle* or simply a *complex center*. The set in the space of parameters of (3) which corresponds to systems having a complex center is the variety $\mathbf{V}(\mathcal{B})$ of the Bautin ideal and it is called the *center variety* of (3). If (3) is the complexification of a real system (1), then going back to the coordinates (u,v) we obtain from Ψ a first integral of (1) and conclude that the real system (1) has a center at the origin. Hence we have a generalization of the center problem to systems with a complex saddle at the origin. As we have mentioned knowing conditions of integrability of a complex system (3) one can easily derive conditions for integrability of real system (1).

The integrability problem for systems (3), where the nonlinearity is a quadratic or cubic polynomial, has been intensively studied, see e.g. [2,5,19,22,26] and references given there. Recently several works have been

[1] We recall that the variety of a polynomial ideal is the set of common zeros of all polynomials of the ideal.