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Dynamic programming principle of control systems on manifolds and its relations to maximum principle ☆

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ABSTRACT

We study the dynamic programming principle (DPP for short) on manifolds, obtain the Hamilton–Jacobi–Bellman (HJB for short) equation, and prove that the value function is the only viscosity solution to the HJB equation. Then, we investigate the relation between DPP and Pontryagin's maximum principle (PMP for short), from which we obtain PMP on manifolds.

1. Introduction

Let M be a complete Riemannian manifold of dimension n. Let ∇ be the Levi-Civita connection of M, $\rho(\cdot,\cdot)$ be the distance function on M, T_xM be the tangent space of the manifold M at $x \in M$, and T_x^*M be the cotangent space. Let $TM = \bigcup\limits_{x\in M} T_xM$ denote the tangent bundle, $T^*M = \bigcup\limits_{x\in M} T_x^*M$ denote the cotangent bundle of M, and $C^\infty(M)$ denote the set of all smooth functions on M.

Let U be a metric space. Let $f : [0,T] \times M \times U \to TM$ be a given map. Given $T > 0$ and $y_0 \in M$, we consider the following control system

$$\begin{cases} \dot{y}(t) = f(t, y(t), u(t)), \quad \text{a.e.}\, t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{1.1}$$

where $\dot{y}(t)$ denotes $\frac{d}{dt}y(t)$ for $t \in [0,T]$, and the control $u(\cdot)$ belongs to

$$\mathcal{U}[0,T] \equiv \{u(\cdot) : [0,T] \to U | u(\cdot) \text{ is measurable}\}.$$

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The cost functional associated with (1.1) is

$$J(u(\cdot)) = \int_0^T f^0(t, y(t), u(t))dt + h(y(T)), \tag{1.2}$$

for some given functions f^0 defined on $[0,T] \times M \times U$ and h defined on M. The optimal control problem is stated as follows:

Problem (D) Minimize (1.2) in u over $\mathcal{U}[0,T]$.

To consider the dynamic programming principle (DPP) for Problem (D), we consider a family of optimal control problems stated as follows.

Let $(t,x) \in [0,T) \times M$. We consider the following control system over $[t,T]$

$$\begin{cases} \dot{y}(s) = f(s, y(s), u(s)), & \text{a.e. } s \in (t,T], \\ y(t) = x, \end{cases} \tag{1.3}$$

where control $u(\cdot)$ belongs to

$$\mathcal{U}[t,T] = \{u(\cdot) : [t,T] \to U; u(\cdot) \text{ is measurable}\}.$$

The solution to (1.3) corresponding to initial time t, initial state x, and control $u(\cdot)$ is denoted by $y(\cdot; t, x, u(\cdot))$. The cost function with respect to (1.3) is

$$J(u(\cdot); t, x) = \int_t^T f^0\Big(\tau, y(\tau; t, x, u(\cdot)), u(\tau)\Big) d\tau + h\Big(y(T; t, x, u(\cdot))\Big). \tag{1.4}$$

The corresponding optimal control problem is

Problem (D_{tx}) Given $(t,x) \in [0,T) \times M$, find $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ such that

$$J(\bar{u}(\cdot); t, x) = \inf_{u(\cdot) \in \mathcal{U}[t,T]} J(u(\cdot); t, x). \tag{1.5}$$

We call the control $\bar{u}(\cdot)$ an optimal control, the corresponding solution $\bar{y}(\cdot)$ the optimal trajectory, and $(\bar{y}(\cdot), \bar{u}(\cdot))$ the optimal pair.

Definition 1.1. Define

$$\begin{cases} V(t,x) = \inf\limits_{u(\cdot) \in \mathcal{U}[t,T]} J(u(\cdot); t, x), & \forall (t,x) \in [0,T) \times M, \\ V(T,x) = h(x), \quad \forall x \in M. \end{cases} \tag{1.6}$$

We call $V(\cdot,\cdot)$ the value function of problem (D).

Problem (D_{tx}) is a family of optimal control problems parameterized by $(t,x) \in [0,T) \times M$, and the original Problem (D) is a special case of Problem (D_{tx}) by taking $t = 0$. The idea of DPP is to dig out the relation between Problem (D) and Problem (D_{tx}) by investigation of the value function. As a result, one can construct an optimal control of Problem (D).

Optimal control problem on a manifold can be viewed as an optimal control problem with state constrained on a submanifold of Euclidean space, whose interior is empty. Optimal control problems with state