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Multiplicity of nonnegative solutions for quasilinear Schrödinger equations

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ABSTRACT

The paper deals with existence and multiplicity of nonnegative weak solutions for quasilinear Schrödinger equations involving nonlinearities with possibly supercritical growth at infinity and indefinite sign. An appropriate change of variables reduces the quasilinear problem into a semilinear one. Variational and sub- super-methods are applied in order to obtain the main results of the paper.

1. Introduction

In this paper we prove existence and multiplicity of nonnegative weak solutions for quasilinear Schrödinger equations involving nonlinearities with possibly supercritical growth at infinity and indefinite sign. More precisely, we consider

$$-\Delta u + V(x)u - \kappa\Delta(u^2)u = p(u), \tag{1.1}$$

in $\Omega \subset \mathbb{R}^N$, with $N \geq 3$, where $V : \Omega \to \mathbb{R}$ is an appropriate potential, κ is a positive constant and $p : \mathbb{R} \to \mathbb{R}$ is a continuous function, with possibly supercritical growth at infinity. We point out that p may change sign.

When $\kappa \neq 0$, to the best of our knowledge, there are few papers leading with supercritical problems. In [17] Moameni covered this case when $\Omega = \mathbb{R}^N$ and obtained the existence of positive solutions assuming, among other conditions, that the nonlinearity p is a nonnegative function, $N = 2$, and the potential function V is

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radial, that is $V(x) = V(|x|)$ for any $x \in \mathbb{R}^N$, and V vanishes in some annulus of $\mathbb{R}^N$. This allowed him to recover the desired compact embedding to prove compactness.

Solutions of (1.1) are closely related to the standing wave solutions of the quasilinear Schrödinger equation

$$i\partial_t z = -\Delta z + W(x)z - f(|z^2|)z - \kappa\Delta\left[g(|z^2|)\right] g'(|z^2|)z, \tag{1.2}$$

where $W = W(x)$, $x \in \mathbb{R}^N$, is a given potential, κ is a real constant, and f, g are continuous real functions. Quasilinear equations of the form (1.2) find applications in several areas of physics, as explained in [3].

It is well known that $z(t,x) = \exp(-iEt)u(x)$ satisfies (1.2) if and only if the function $u = u(x)$ solves the elliptic equation (1.1), when $V(x) = W(x) - E$ is the new potential. When $p(u) = |u|^{r-1}u$, $u \in \mathbb{R}$, $r + 1 = 2 \cdot 2^* = 4N/(N-2)$, and $N \geq 3$, then p is critical for the equation (1.1). We refer to Remark 3.13 of [11] for further details concerning the critical case. The subcritical case $r+1 < 2\cdot 2^*$ was first described in the paper [18], which was then extended in [10]. Several papers are devoted to (1.1) in critical or subcritical case. We just mention [5,8,12,21,16,23,24,13,14] and the references therein.

More specifically, in this paper we consider the problem

$$\begin{cases} -\Delta u - \Delta(u^2)u = \lambda u + k(x)\,|u|^{q-1}\,u - h(x)\,|u|^{r-1}\,u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where Ω is a bounded domain, with smooth boundary $\partial\Omega$, of $\mathbb{R}^N$, $N \geq 3$. The exponent q satisfies the restriction $3 \leq q < r < \infty$, and the functions h, k are nonnegative and of class $L^\infty(\Omega)$. Problem (P_λ) has already been studied in the recent paper [15] under the main assumption

$$\int_{\Omega_2} k(x)\left[\frac{k(x)}{h(x)}\right]^{(q+1)/(r-q)} < \infty, \quad \text{where } \ \Omega_2 = \operatorname{supp} k. \tag{1.3}$$

The main goal of the paper is to complete the picture given in [15], extending the key request (1.3) into

$$\int_{\Omega_2} \left[\frac{k(x)^{(r-1)/(r-q)}}{h(x)^{(q-1)/(r-q)}}\right]^{N/2} < \infty. \tag{1.4}$$

For different quasilinear problems in the entire $\mathbb{R}^N$ condition (1.4) has already been required in [20], while (1.3) in [2,19,20]. Assumptions (1.3) and (1.4) first appear in the famous paper [1] devoted to the study of semilinear elliptic Dirichlet problems.

It is worth mentioning that (1.3) implies (1.4), when q is subcritical. However, we emphasize that the proof techniques we use are different than those adopted in [1,2,19,15,20], since (1.4) makes (P_λ) fairly delicate to treat and new strategies must be found.

Throughout the paper, we assume

(H_1) $\operatorname{supp} k \subset \operatorname{supp} h$,
(H_2) $h, k \in L^\infty(\Omega)$ *are nonnegative functions, and* $\operatorname{supp} k$ and $\operatorname{supp} h$ *have positive measure,*

and adopt the following notations

- $\Omega_1 = \operatorname{supp} h$, $\Omega_2 = \operatorname{supp} k$;
- $\widetilde{\Omega} = \{x \in \Omega : h(x) = 0\}$;
- In all the integrals we omit the symbol "dx";