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A new topological degree theory for pseudomonotone perturbations of the sum of two maximal monotone operators and applications

Teffera M. Asfaw

Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA

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ABSTRACT

Let X be a real reflexive locally uniformly convex Banach space with locally uniformly convex dual space X^* and G be a nonempty, bounded and open subset of X. Let $T : X \supseteq D(T) \to 2^{X^*}$ and $A : X \supseteq D(A) \to 2^{X^*}$ be maximal monotone operators. Assume, further, that, for each $y \in X$, there exists a real number $\beta(y)$ and there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$, $\phi(t) \to \infty$ as $t \to \infty$ satisfying

$$\langle w^*, x - y\rangle \geq -\phi(\|x\|)\|x\| - \beta(y)$$

for all $x \in D(A)$, $w^* \in Ax$, and $S : X \to 2^{X^*}$ is bounded of type (S_+) or bounded pseudomonotone such that $0 \notin (T + A + S)(D(T) \cap D(A) \cap \partial G)$ or $0 \notin \overline{(T + A + S)(D(T) \cap D(A) \cap \partial G)}$, respectively. New degree theory is developed for operators of the type $T+A+S$ with degree mapping $d(T+A+S, G, 0)$. The degree is shown to be unique invariant under suitable homotopies. The theory developed herein generalizes the Asfaw and Kartsatos degree theory for operators of the type $T+S$. New results on surjectivity and solvability of variational inequality problems are obtained. The mapping theorems extend the corresponding results for operators of type $T+S$. The degree theory developed herein is used to show existence of weak solution of nonlinear parabolic problem in appropriate Sobolev spaces.

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1. Introduction and preliminaries

Throughout the paper, $(X, \|\cdot\|)$ denotes a real reflexive locally uniformly convex Banach space with locally uniformly convex dual space X^*. For $x \in X$ and $x^* \in X^*$, the duality pairing $\langle x^*, x\rangle$ denotes the value $x^*(x)$. Let $J : X \to 2^{X^*}$ be the normalized duality mapping given by

E-mail address: teffera6@vt.edu.

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$$J(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2,\ \|x^*\| = \|x\|\}.$$

It is well-known that, for each $x \in X$, $J(x) \neq \emptyset$. Since X and X^* are locally uniformly convex, J is single valued and homeomorphism. For a multivalued operator T from X into X^*, the domain of T, denoted by $D(T)$, is given as $D(T) = \{x \in X : Tx \neq \emptyset\}$. The range of T, denoted by $R(T)$, is given by $R(T) = \cup_{x\in D(T)} Tx$ and graph of T, denoted by $G(T)$, is given by $G(T) = \{(x, v^*) : x \in D(T), v^* \in Tx\}$. A multivalued operator $T : X \supseteq D(T) \to 2^{X^*}$ is called "monotone" if for all $x \in D(T)$, $y \in D(T)$, $v^* \in Tx$ and $u^* \in Ty$, we have

$$\langle v^* - u^*, x - y\rangle \geq 0.$$

It is called "maximal monotone" if $R(T + \lambda J) = X^*$ for every $\lambda > 0$, i.e., T is maximal monotone if and only if T is monotone and $\langle u^* - u_0^*, x - x_0\rangle \geq 0$ for every $(x, u^*) \in G(T)$ implies $x_0 \in D(T)$ and $u_0^* \in Tx_0$. Let $T : X \supseteq D(T) \to 2^{X^*}$ be maximal monotone. For each $t > 0$, the operators $T_t : X \to X^*$ and $J_t : X \to D(T)$ defined by $T_t x = (T^{-1} + tJ^{-1})^{-1}x$ and $J_t x = x - tJ^{-1}(T_t x)$, $x \in X$, are called the Yosida approximant and resolvent of T, respectively. For each $t > 0$, T_t is continuous maximal monotone and J_t is continuous such that $T_t x \in T(J_t x)$ for all $x \in X$. Furthermore, for each $x \in D(T)$, we have $\|T_t x\| \leq |Tx|$ uniformly for all $t > 0$, where $|Tx| = \inf\{\|y^*\| : y^* \in Tx\}$. A mapping $T : X \supseteq D(T) \to 2^{X^*}$ is called "quasibounded" if for every $M > 0$, there exists $K(M) > 0$ such that $[x, w^*] \in G(T)$ with $\|x\| \leq M$ and $\langle w^*, x\rangle \leq M\|x\|$ imply $\|w^*\| \leq K(M)$. It is "strongly quasibounded" if for every $M > 0$, there exists $K(M) > 0$ such that $[x, w^*] \in G(T)$ with $\|x\| \leq M$ and $\langle w^*, x\rangle \leq M$ imply $\|w^*\| \leq K(M)$. A mapping $A : X \supset D(A) \to 2^{X^*}$ is "coercive" if either $D(A)$ is bounded or there exists a function $\psi : [0, \infty) \to (-\infty, \infty)$ such that $\psi(t) \to \infty$ as $t \to \infty$ and

$$\langle y^*, x\rangle \geq \psi(\|x\|)\|x\| \text{ for all } x \in D(A) \text{ and } y^* \in Ax.$$

It is called "weakly coercive" if either $D(A)$ is bounded or $|Ax| \to \infty$ as $\|x\| \to \infty$, where for each $x \in D(A)$,

$$|Ax| = \inf\{\|v^*\| : v^* \in Ax\}.$$

The theory of single-valued pseudomonotone operators is due to Brézis [15]. Browder and Hess [22] introduced the following definition of multi-valued pseudomonotone and generalized pseudomonotone operators.

Definition 1. An operator $T : X \supset D(T) \to 2^{X^*}$ is said to be

(a) "pseudomonotone" if the following conditions are satisfied.
  (i) For every $x \in D(T)$, Tx is nonempty, closed, convex and bounded subset of X^*;
  (ii) T is finitely continuous, i.e., T is "weakly upper semicontinuous" on each finite-dimensional subspace F of X, i.e., for every $x_0 \in D(T) \cap F$ and every weak neighborhood V of Tx_0 in X^*, there exists a neighborhood U of x_0 in F such that $TU \subset V$;
  (iii) For every sequence $\{x_n\} \subset D(T)$ and every sequence $\{y_n^*\}$ with $y_n^* \in Tx_n$ such that $x_n \rightharpoonup x_0 \in D(T)$ and

$$\limsup_{n\to\infty} \langle y_n^*, x_n - x_0\rangle \leq 0,$$

we have that for every $x \in D(T)$, there exists $y^*(x) \in Tx_0$ such that

$$\langle y^*(x), x_0 - x\rangle \leq \liminf_{n\to\infty} \langle y_n^*, x_n - x\rangle.$$