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Limit cycles for 3-monomial differential equations

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ABSTRACT

We study planar polynomial differential equations that in complex coordinates write as $\dot{z} = Az + Bz^k\bar{z}^l + Cz^m\bar{z}^n$. We prove that for each $p \in \mathbb{N}$ there are differential equations of this type having at least p limit cycles. Moreover, for the particular case $\dot{z} = Az + B\bar{z} + Cz^m\bar{z}^n$, which has homogeneous nonlinearities, we show examples with several limit cycles and give a condition that ensures uniqueness and hyperbolicity of the limit cycle.

1. Introduction and main results

The celebrated second part of the Hilbert's 16th problem [17] consists in determining a uniform upper bound on the number of limit cycles of all polynomial differential systems of degree N, see for instance [19,21] and the references therein. This problem is still open even for the quadratic case, $N = 2$. Due to its extreme difficulty, usually people fix a subclass of polynomial differential equations, namely: quadratic, cubic, Kukles, Liénard, homogeneous nonlinearities,..., and then try to advance in the question restricted to the selected family. This paper goes in a similar direction, we consider a simple class of polynomial systems, but instead of fixing the degree, we fix a short number of monomials once the system is written in complex coordinates, and then we study its number of limit cycles.

To be more precise, consider two dimensional real differential systems,

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \quad (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R},$$

with P and Q polynomials. They can also be written as

$$\frac{dz}{dt} = \dot{z} = F(z, \bar{z}), \quad z \in \mathbb{C}, \quad t \in \mathbb{R},$$

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where F is a complex polynomial. In this paper we face the question of the number of limit cycles for polynomial differential equations with three monomials that write as

$$\dot{z} = Az + Bz^k\bar{z}^l + Cz^m\bar{z}^n, \tag{1}$$

where $A, B, C \in \mathbb{C}$ and $k, l, m, n \in \mathbb{N} \cup \{0\}$. Our first result is:

Theorem A. *For any $p \in \mathbb{N}$ there is a differential equation of type (1) having at least p limit cycles.*

A celebrated family of differential equations of the form (1) is

$$\dot{z} = Az + Bz^2\bar{z} + C\bar{z}^{q-1}, \tag{2}$$

with $q \geq 3$. It gives the *versal deformation* of a principal singular smooth systems having rotational invariance of $2\pi/q$ radians, see [3]. The cases $q = 3, 4$ are called *strong resonances* while the cases $q \geq 5$ are called *weak resonances*. The situation $q \neq 4$ is well understood and considered in several places, see for instance [8,18,25] for $q = 3$ and again [3] for $q \geq 5$.

The study of the limit cycles for case $q = 4$ turns out to be especially difficult and is considered in many works, see for instance [1,3,6,9,16,20,22,26]. To know the number of limit cycles surrounding the origin, and eventually surrounding also the other 4 or 8 critical points that the system can posses is yet an open question. In [9,22] it is proved that at least two limit cycles can exist surrounding the 9 critical points.

The problem of the number of limit cycles not surrounding the origin for (2) and $q = 4$ is totally solved. In [26] it is proved that either there are no limit cycles or that there are exactly four ($q = 4$) hyperbolic ones, each one of them surrounding exactly one of the critical points of index +1. It also remains to study the coexistence of these four limit cycles with other limit cycles that surround the origin.

Inspired by the presence of the four limit cycles not surrounding the origin for (2) and $q = 4$ and for the results presented in [21, Sec. 7], we consider a variation of (2) that allow us to prove that for each $p \geq 3$ there are systems in (1) with at least p limit cycles not surrounding the origin. More concretely, our proof of Theorem A relies on the study of the following subclass of (1),

$$\dot{z} = A\,z + B\,z^{p-1}\bar{z}^{p-2} + C\,\bar{z}^{p-1} = A\,z + B\,z|z|^{2(p-2)} + C\,\bar{z}^{p-1}, \tag{3}$$

with $p \geq 3$, which also has rotational invariance of $2\pi/p$ radians. We consider a Hamiltonian case in (3) and we perturb it without leaving the family. Then, studying the quotient of two Abelian integrals associated to the given perturbation, we prove the existence of at least one limit cycle surrounding a critical point that is not the origin. Afterwards, the rotational invariance property provides the existence of the p limit cycles. The main difficulty and difference with similar previous results is that one of the parameters of the differential equation is related with its degree.

Notice that (3) coincides with (2) only when $p = q = 3$. Therefore, for this case our proof gives three limit cycles surrounding three different critical points which posses $2\pi/3$ rotational symmetry. These limit cycles are not showed in [8,9] but already appear in [25].

We remark that when $q \geq 5$ the existence of examples with q limit cycles not surrounding the origin of (2) is no more true, see Lemma 2.4 and Remark 2.5. For this reason, to prove our result we have used (3) instead of (2).

Theorem A shows that there is no upper bound for the number of limit cycles for general systems with three complex monomials. Hence, in the second part of the paper we fix another concrete subfamily with three monomials and we give conditions over its parameters in order to have uniqueness and hyperbolicity of its limit cycles.