J. Math. Anal. Appl. 428 (2015) 838–854

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Blow-up and asymptotic behavior of solutions for reaction–diffusion equations with free boundaries ☆

Ningkui Sun

Department of Mathematics, Tongji University, Shanghai 200092, China

ARTICLE INFO

Article history:
Received 10 February 2015
Available online 24 March 2015
Submitted by Y. Du

Keywords:
Nonlinear diffusion equation
Free boundary problem
Asymptotic behavior
Blow-up
Threshold value

ABSTRACT

We study the blow-up phenomena and the asymptotic behavior of time-global solutions for reaction–diffusion equations $u_t = u_{xx} + au + u^p$ ($a \in \mathbb{R}$ and $p > 1$), with free boundaries and initial data $\sigma\phi(x)$. We give a sharp threshold value $\sigma^* = \sigma^*(\phi, a, p) \geq 0$ such that the solution blows up in finite time when $\sigma > \sigma^*$, vanishes (i.e. $u \to 0$ as $t \to \infty$) when $\sigma < \sigma^*$, and when $\sigma = \sigma^*$, it converges as $t \to \infty$ to 0 or to an evenly decreasing positive stationary solution, depending on whether $a \geq 0$ or $a < 0$. Moreover, when blow-up happens, we show that the blow-up set is compact in the occupying domain of initial data and the free boundaries keep bounded.

1. Introduction

In this paper, we consider the following free boundary problem

$$\begin{cases} u_t = u_{xx} + au + u^p, & g(t) < x < h(t),\ t > 0, \\ u(t, g(t)) = 0,\ g'(t) = -\mu\, u_x(t, g(t)), & t > 0, \\ u(t, h(t)) = 0,\ h'(t) = -\mu\, u_x(t, h(t)), & t > 0, \\ -g(0) = h(0) = h_0,\quad u(0, x) = u_0(x), & -h_0 \leq x \leq h_0, \end{cases} \tag{1.1}$$

where $a \in \mathbb{R}$ and $p > 1$, $x = g(t)$ and $x = h(t)$ are the moving boundaries to be determined together with $u(t, x)$, μ is a given positive constant. The initial function u_0 belongs to $\mathscr{X}(h_0)$ for some $h_0 > 0$, where

$$\mathscr{X}(h_0) := \Big\{ \phi \in C^2([-h_0, h_0]) : \phi(-h_0) = \phi(h_0) = 0,\ \phi'(-h_0) > 0, \\ \phi'(h_0) < 0,\ \phi(x) > 0 \text{ in } (-h_0, h_0) \Big\}. \tag{1.2}$$

☆ This research was partly supported by the NSFC (11271285).
E-mail address: sunnk1987@163.com.

http://dx.doi.org/10.1016/j.jmaa.2015.03.058

The main purpose of this paper is to classify the blow-up phenomena and the long time behavior of solutions of (1.1) for all $a \in \mathbb{R}$ and $p > 1$.

It is known that a lot of physical and biological phenomena can be described by the following reaction–diffusion equation:

$$u_t(t,x) = u_{xx}(t,x) + F(u(t,x)), \tag{1.3}$$

where the variable $u(t,x)$ can be seen as the temperature in a chemical reaction or the population density of a biological species (see [23]), the nonlinearity F stands for a net heat source (i.e. the natural heat source subtract the dissipative heat source) or a net birth rate (i.e. the natural birth rate subtract the natural death rate), which satisfies some growth conditions: it becomes positive when u is large enough, or is positive for all $u > 0$. The second order derivative u_{xx} represents the diffusion. In a chemical reaction, we know that the chemical reaction will generate heat and the high temperature will accelerate the chemical reaction, thus the temperature will become higher and higher. Therefore, if the initial temperature is high enough, then the temperature will likely become *very high* in finite time, which is called a *blow-up phenomenon*. Similarly, in a biological species, the large population density will bring the high birth rate, on the other hand, the high birth rate will accelerate the population growth. Hence, if the initial population density is large enough, then blow-up will happen, too.

Motivated by such phenomena arising in various applied fields, the blow-up problem for Eq. (1.3) has been studied by many authors (cf. [11–24] and references therein). For example, the classical papers [12,16] considered the Cauchy problem

$$u_t = u_{xx} + au + u^p \ (x \in \mathbb{R},\ t > 0),\ u(0,x) = u_0(x) \geq 0 \ (x \in \mathbb{R}) \tag{1.4}$$

with $a = 0$ and proved that, when $1 < p \leq 3$, a nontrivial solution must blow up in finite time; while, when $p > 3$, problem (1.4) has time-global solutions for some small initial data. In [3,4,21] the authors studied the problem (1.4) with $a < 0$ and gave a trichotomy result: any solution either blows up in finite time, or vanishes (i.e., $u \to 0$ as $t \to \infty$), or converges to an evenly decreasing positive steady state as $t \to \infty$. The author in [17] studied the corresponding steady state problem on annulus with $a < 0$. Moreover in [20], the author studied the corresponding steady state problem of (1.4) with $a \geq 0$ and established some existence and nonexistence results. In this paper, we consider the problem with two free boundaries as in (1.1), and prove trichotomy results for any $a \in \mathbb{R}$ and $p > 1$. In particular, our results imply that problem (1.1) may have time-global solution even if $a = 0$ and $1 < p \leq 3$ provided that the initial data is small, which is different from the results in [12,16].

Problem (1.1) with other type of nonlinearities, such as monostable, bistable, combustion or degenerate Fisher–KPP type of nonlinearities was studied recently in [5–7,25]. They used the free boundary problem to describe the spreading of a new or invasive species in which the free boundaries $x = g(t)$ and $x = h(t)$ represent the spreading fronts of the population whose density is represented by $u(t,x)$ (for more background, see [2,18,22]). In their models, all the solutions exist globally. They gave a rather complete description on the long-time dynamical behavior of the solutions. In this paper we adopt a nonlinearity as the form of $au + u^p$ with $a \in \mathbb{R}$ and $p > 1$. In this case blow-up can happen in finite time as in the Cauchy problems, which is different from the cases with monostable or bistable nonlinearities. Some special cases of problem (1.1) have been studied in the last decades. For example, the authors in [10,13,26] studied problem (1.1) with $a = 0$. They showed some conditions which can imply blow-up and the existence of time-global solutions. Moreover, they obtained that all time-global solutions are bounded. For more references of blow-up problems with free boundary condition, please see [1].

In our paper, inspired by the results mentioned above, we want to present a complete description on the blow-up phenomena and long-time dynamical behavior of time-global solutions of (1.1) for all $a \in \mathbb{R}$ and $p > 1$.