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Distributional chaos in dendritic and circular Julia sets

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ABSTRACT

If x and y belong to a metric space X, we call (x, y) a DC1 scrambled pair for $f : X \to X$ if the following conditions hold:

1) for all $t > 0$, $\limsup_{n\to\infty} \frac{1}{n} \left|\{0 \le i < n : d(f^i(x), f^i(y)) < t\}\right| = 1$, and
2) for some $t > 0$, $\liminf_{n\to\infty} \frac{1}{n} \left|\{0 \le i < n : d(f^i(x), f^i(y)) < t\}\right| = 0$.

If $D \subset X$ is an uncountable set such that every $x, y \in D$ form a DC1 scrambled pair for f, we say f exhibits *distributional chaos of type 1*. If there exists $t > 0$ such that condition 2) holds for any distinct points $x, y \in D$, then the chaos is said to be *uniform*. A *dendrite* is a locally connected, uniquely arcwise connected, compact metric space. In this paper we show that a certain family of quadratic Julia sets (one that contains all the quadratic Julia sets which are dendrites and many others which contain circles) has uniform DC1 chaos.

1. Introduction

In [10], B. Schweizer and J. Smítal introduced a notion of chaos which has been generalized to three types, known as distributional chaos of types 1, 2, and 3. In this paper we focus on the strongest of these properties, distributional chaos of type 1 (DC1), which is also referred to as *Schweizer–Smítal chaos.*

In [5, Theorem 1.1], Downarowicz proved that if a topological dynamical system (X, f) has positive topological entropy $h_{top}(f) > 0$, then the system possesses an uncountable DC2-scrambled set. As quadratic maps on the complex plane have positive topological entropy on their Julia sets [6], it is thus clear that every quadratic Julia set dendrite map exhibits DC2. We show that many quadratic Julia sets in fact exhibit uniform DC1.

We take a symbolic approach to this problem. We focus specifically on two families of abstract Julia sets introduced by Baldwin [2]. The first family contains all the dendritic Julia sets, and each such Julia set

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is a shift-invariant subspace of a non-Hausdorff itinerary space which is called Λ. We also consider certain quadratic Julia sets generated by maps with an attracting or parabolic periodic point. These are sometimes called "circular" Julia sets because they contain many circles. These Julia sets are naturally represented in a non-Hausdorff itinerary space which we call Γ. The itinerary topology on Λ mimics in many ways the itinerary topology generated by a unimodal map on $[0,1]$, while the topology on Γ is similar to the itinerary topology generated by angle-doubling on the unit circle.

In the next section we give the necessary background definitions for the topology on Λ and for a description of the abstract Julia sets in Λ that contain conjugate copies of every quadratic Julia set which is a dendrite. In Section 3, we use this symbolic representation to prove that these systems have distributional chaos of type 1. In Section 4, we introduce the itinerary space Γ and the abstract "circular" Julia sets, and we prove that these systems also have DC1.

2. Preliminaries

In this section we introduce the symbolic representation of Λ and the dendritic Julia sets due to Baldwin [1] and [2]. This is a natural extension of the kneading theory for unimodal maps on $[0,1]$.

Let σ represent the usual one-sided shift map. Let $\omega = \mathbb{N} \cup \{0\}$ and let Λ be the product space on $\{0,1,*\}^\omega$, where each factor space $\{0,1,*\}$ has topology induced by the basis $\{\{0\},\{1\},\{0,1,*\}\}$. We then see that Λ is a non-Hausdorff space since if two points in Λ disagree only where one has a $*$, the points cannot be separated with open sets. This is consistent with letting 0 and 1 represent open regions, S_0 and S_1, in the dendrite while $*$ represents the unique critical point in $\overline{S_0} \cap \overline{S_1}$.

Definition 1. A sequence $\tau \in \Lambda$ is called Λ-*acceptable* if and only if

1) for all $n \in \omega$, we have $\tau_n = *$ if and only if $\sigma^{n+1}(\tau) = \tau$, and
2) for all $n \in \omega$ such that $\sigma^n(\tau) \neq \tau$ there exists $m \in \omega$ such that $* \neq \tau_{m+n} \neq \tau_m \neq *$.

Condition 1) means that either τ is periodic, where $*$ signals the end of the period, or τ is not periodic and $\tau_n \neq *$ for all $n \in \omega$ (though possibly τ is eventually periodic). Condition 2) means that if $\sigma^n(\tau) \neq \tau$, then $\sigma^n(\tau)$ and τ can be separated by open sets – in other words, they differ in a place where neither is $*$. Λ-acceptable sequences are the possible kneading sequences.

Definition 2. If τ is Λ-acceptable, then $x \in \Lambda$ is (Λ,τ)-*consistent* if and only if for all $n \in \omega$, $x_n = *$ implies $\sigma^{n+1}(x) = \tau$.

Note that $\alpha \in \{0,1\}^\omega$ is (Λ,τ)-consistent for any Λ-acceptable τ.

Definition 3. The point $x \in \Lambda$ is (Λ,τ)-*admissible* if and only if it is (Λ,τ)-consistent and for all $n \in \omega$ such that $\sigma^n(x) \neq *\tau$, there exists $m > 0$ such that $* \neq x_{m+n} \neq \tau_{m-1} \neq *$ (that is, there is a position where $\sigma^n(x)$ and $*\tau$ differ and neither is a $*$). If τ is Λ-acceptable, let $\mathcal{D}_\tau = \{x \in \Lambda : x \text{ is } (\Lambda,\tau)\text{-admissible}\}$.

A *dendrite* is a locally connected, uniquely arcwise connected, compact metric space. The following two theorems are due to Baldwin:

Theorem 4. *(See [1].) Let τ be Λ-acceptable. Then $\mathcal{D}_\tau$ is a shift-invariant self-similar dendrite.*

Theorem 5. *(See [2, Theorem 2.5].) Let $f_c(z) = z^2 + c$. If J_c is a dendrite, then there is a Λ-acceptable τ such that $f_c|J_c$ is conjugate to $\sigma|\mathcal{D}_\tau$.*