J. Math. Anal. Appl. 428 (2015) 989–1004

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Porosity results on fixed points for nonexpansive set-valued maps in hyperbolic spaces

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ARTICLE INFO

Article history:
Received 13 December 2014
Available online 24 March 2015
Submitted by A. Daniilidis

Keywords:
Hyperbolic space
Porosity
Fixed point
Nonexpansive set-valued map
Topological structure

ABSTRACT

We establish the porosity results on the existence and the topological structure of fixed points for nonexpansive set-valued maps in hyperbolic spaces, which in particular extend and/or improve some known results in this direction.

1. Introduction

The study of the existence problem of fixed points for maps in geodesic space dates back at least to the works due to Kirk in [20,21], where it was proved that every nonexpansive (single-valued) map defined on a bounded closed convex subset of a complete CAT(0) space has a fixed point. This existence result was followed by a series of new works by different authors, see for example [1–3,7,8,13,22,23,25,34], mainly focusing on CAT(0) space and on extending Kirk's results from the single-valued case to the set-valued case. For example, Dhompongsa et al. in [13] showed that a nonexpansive compact-valued map defined on a convex subset, which satisfies the "weakly inward condition", has a fixed point; while, in an R-tree, the special CAT(0) space, Markin in [23] showed that a set-valued "generalized" nonexpansive map has a fixed point. Other extensions from CAT(0) space to CAT(κ) space can be found in [16].

As is known, for a set-valued map, in the Banach space setting (even for a single-valued maps), the fixed point set is not necessarily nonempty. Thus it makes sense to explore if the set of nonexpansive (single-valued

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http://dx.doi.org/10.1016/j.jmaa.2015.03.055

or set-valued) maps that have fixed points is generic. In the Banach space setting, De Blasi and Majak showed in [10] that the set of all single-valued nonexpansive maps without fixed point is σ-porous; this result was extended in [12,26,31,32] to the case of nonexpansive compact-valued maps. However, it seems little extensions to be known in general geodesic spaces except the work due to Reich and Zaslavski in [30], where the porosity property on the existence of fixed points for single-valued maps in hyperbolic spaces was established.

Another interesting topic concerning nonexpansive set-valued maps is the topological structure of fixed point sets because, unlike in the single-valued case (cf. [33]), the fixed point set for a nonexpansive set-valued map (even for strictly contractive maps) is not necessarily a singleton (if it is nonempty). In the Banach space setting, De Blasi and Myjak et al. proved in [12] that the fixed point set of most (in the sense of Baire catalog) convex and compact-valued nonexpansive maps on a closed, bounded and convex subset with nonempty interior is a nonempty and compact R_δ-set; this result was improved in [26] by showing that the set of all convex and compact-valued nonexpansive maps, for which the fixed point set fails to be a nonempty R_δ-set, is σ-porous.

The purpose of the present paper is to study the porosity properties on the existence and the topological structure of fixed points for nonexpansive set-valued maps in hyperbolic spaces, which contains Banach spaces, CAT(0) spaces, in particular, all Hadamard manifolds, and the Hilbert ball [17] as special cases (see Remark 2.1). The main results are stated in Theorems 3.1 and 4.2, which respectively show that the set of all nonexpansive compact set-valued maps (from a star-shaped set to an admissible family) without fixed points is σ-porous and that the set of all nonexpansive convex and compact set-valued maps on a bounded closed convex subset, for which the fixed point set fails to be a nonempty R_δ-set, is σ-porous. These results extend and/or improve the corresponding results in [10,12,26,30]. In particular, Corollary 4.1 improves [12, Theorem 3.1] and [26, Theorem 3] by removing the nonempty interior assumption; while Theorem 4.2 seems new, even in the case when E is a CAT(0) space or Hilbert ball.

The paper is organized as follows. Section 2 contains notations, terminology and lemmas which will be used later. In Sections 3 and 4, the porosity properties on the existence and the topological structure of fixed points for nonexpansive set-valued maps are presented respectively.

2. Preliminaries

Let (E, d) be a metric space. A geodesic in E is an isometry from $\mathbb{R}$ into E (we may also refer to the image of isometry as a geodesic). Let $x, y \in E$. A geodesic joining x to y is a map $\gamma : [0, l] \to E$, where $[0, l] \subseteq \mathbb{R}$, such that $\gamma(0) = x$, $\gamma(l) = y$ and $\mathrm{d}(\gamma(t), \gamma(t')) = |t - t'|$ for $t, t' \in [0, l]$, and the image $\gamma([0, l])$ of γ forms a geodesic segment joining x to y. Note that the geodesic segment joining x to y is not necessarily unique. The space (E, d) is called a geodesic space if each pair of two points of E are joined by a geodesic segment. Let $A \subseteq E$ be a bounded subset. We use $\overline{A}$ and $\operatorname{diam} A$ to denote the closure and diameter of A, respectively; while the distance function associated to A is defined by

$$\mathrm{d}(x, A) := \inf_{a \in A} \mathrm{d}(x, a), \quad \forall x \in E. \tag{2.1}$$

For any $r > 0$, we use $\mathbf{B}(A, r)$ and $\mathbf{U}(A, r)$ to denote the set of all $y \in E$ such that $\mathrm{d}(y, A) \le r$ and $\mathrm{d}(y, A) < r$ respectively. That is,

$$\mathbf{B}(A, r) := \{y \in E : \mathrm{d}(y, A) \le r\}; \ \mathbf{U}(A, r) := \{y \in E : \mathrm{d}(y, A) < r\}. \tag{2.2}$$

In particular, in the special case when $A = \{a\}$, we write $\mathbf{B}(a, r)$ and $\mathbf{U}(a, r)$ for $\mathbf{B}(A, r)$ and $\mathbf{U}(A, r)$ to denote the closed and open ball with center a and radius r, respectively. Let Λ denote the set of all geodesic segments in E. Definitions 2.1 and 2.2 below are taken from [4] and [29], respectively.