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Projective structures in loop quantum cosmology

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A R T I C L E I N F O

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A B S T R A C T

Projective structures have successfully been used for the construction of measures in the framework of loop quantum gravity. In the present paper, we establish such structures for the configuration space $\mathbb{R}\sqcup\mathbb{R}_{\mathrm{Bohr}}$, recently introduced in the context of homogeneous isotropic loop quantum cosmology. In contrast to the traditional space $\mathbb{R}_{\mathrm{Bohr}}$, the first one is canonically embedded into the quantum configuration space of the full theory. In particular, for the embedding of states into a corresponding symmetric sector of loop quantum gravity, this is advantageous. However, in contrast to the traditional space, there is no Haar measure on $\mathbb{R}\sqcup\mathbb{R}_{\mathrm{Bohr}}$ defining a canonical kinematical L^2-Hilbert space on which operators can be represented. The introduced projective structures allow to construct a family of natural measures on $\mathbb{R}\sqcup\mathbb{R}_{\mathrm{Bohr}}$ whose corresponding L^2-Hilbert spaces we finally investigate.

1. Introduction

In the framework of loop quantum gravity (LQG), measures are usually constructed by means of projective structures on the quantum configuration space of interest. For instance, the Ashtekar–Lewandowski measure arises in this way [3], and the same is true for the Haar measure on the Bohr compactification $\mathbb{R}_{\mathrm{Bohr}}$ of $\mathbb{R}$ [16]. This has been used as quantum configuration space for homogeneous isotropic loop quantum cosmology (LQC); a symmetry reduced version of LQG designed to describe the early universe near the Big Bang [4]. Unfortunately, there is no continuous embedding of $\mathbb{R}_{\mathrm{Bohr}}$ into the quantum configuration space of LQG which additionally extends the embedding of the respective reduced classical configuration space [7]. This property, however, is crucial for the embedding approach for states formulated in [5].

Now, non-embeddability arises from the fact that, in contrast to the full theory, the cosmological quantum configuration space has been defined by means of linear curves instead of all the embedded analytic ones [7]. Thus, to overcome this problem, in [11] the embedded analytic curves were used to define the reduced quantum space as well; now being given by $\overline{\mathbb{R}} = \mathbb{R}\sqcup\mathbb{R}_{\mathrm{Bohr}}$. In particular, the embedding approach from [5] here can be applied once a reasonable measure has been fixed. Now, since no Haar measure exists on $\overline{\mathbb{R}}$ [13], such measures have to be constructed by hand.

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In the present paper, we attack this issue by means of projective structures on $\overline{\mathbb{R}}$ which we then use to motivate the family of normalized Radon measures

$$\mu_{\rho,t}(A) = t \cdot \rho(\lambda)(A \cap \mathbb{R}) + (1-t) \cdot \mu_{\mathrm{Bohr}}(A \cap \mathbb{R}_{\mathrm{Bohr}}) \qquad \forall\, A \in \mathfrak{B}(\overline{\mathbb{R}}) \tag{1}$$

for $0 \leq t \leq 1$ and $\rho(\lambda)$ the push forward of the Lebesgue measure λ on $(0,1)$ by some homeomorphism $\rho\colon (0,1) \to \mathbb{R}$. For this, we first reformulate the definition of a projective limit in a way more practicable for defining measures on compact Hausdorff spaces, such as, e.g., $\overline{\mathbb{R}}$. Then, we motivate a certain family of projective structures which will provide us with the measures (1). As we will see, these measures give rise to only two different Hilbert space structures on $\overline{\mathbb{R}}$. More precisely, up to *canonical* isomorphisms, we will have the following three cases:

$$L^2(\mathbb{R},\lambda), \qquad L^2(\mathbb{R},\lambda) \oplus L^2(\mathbb{R}_{\mathrm{Bohr}},\mu_{\mathrm{Bohr}}), \qquad L^2(\mathbb{R}_{\mathrm{Bohr}},\mu_{\mathrm{Bohr}}),$$

whereby $L^2(\mathbb{R},\lambda) \oplus L^2(\mathbb{R}_{\mathrm{Bohr}},\mu_{\mathrm{Bohr}})$ and $L^2(\mathbb{R}_{\mathrm{Bohr}},\mu_{\mathrm{Bohr}})$ are isometrically isomorphic, just by dimensional arguments. Anyhow, since $L^2(\mathbb{R},\lambda)$ is separable and $L^2(\mathbb{R}_{\mathrm{Bohr}},\mu_{\mathrm{Bohr}})$ is not so, there cannot exist any isometric isomorphism between these two spaces.

This paper is organized as follows:

- ▹ In Section 2, we fix the notations and provide a characterization of projective limits convenient for defining measures. In Section 3, we briefly review some facts on invariant homomorphisms [13] that we will need in the main part of this paper.
- ▹ In Section 4, we first discuss some elementary properties of the space $\overline{\mathbb{R}}$. In particular, we prove a uniqueness result concerning the assumptions made in [9] to the inner product on $C_0(\mathbb{R}) \oplus C_{\mathrm{AP}}(\mathbb{R})$. Then, we investigate how to write $\overline{\mathbb{R}}$ as projective limit, in order to construct reasonable Radon measures thereon. Here, we discuss several possibilities, finally leading to the projective structures presented in the third part of Section 4. Basically, there we will use the fact that for each nowhere vanishing[1] $f \in C_0(\mathbb{R})$ the functions $\{f\} \sqcup \{\chi_l\}_{l\in\mathbb{R}}$ with $\chi_l\colon x \mapsto \mathrm{e}^{\mathrm{i}lx}$ generate the C^*-algebra $C_0(\mathbb{R}) \oplus C_{\mathrm{AP}}(\mathbb{R})$. Then, each such f which is in addition injective will give rise to a projective structure similar to that one introduced in [16] for the space $\mathbb{R}_{\mathrm{Bohr}}$. In the last part, we finally use these structures to construct a family of normalized Radon measures on $\overline{\mathbb{R}}$ which we then show to define two different non-isomorphic L^2-Hilbert spaces on $\overline{\mathbb{R}}$.

2. Preliminaries

We start this section by fixing the notations. Then, we give a short introduction into projective structures on compact Hausdorff spaces and consistent families of normalized Radon measures.

2.1. Notations

A curve γ in a manifold M is a continuous map $\gamma\colon I \to M$ for $I \subseteq \mathbb{R}$ an interval, i.e., of the form $[a,b], [a,b), (a,b]$ or (a,b) for $a < b$. Then, the curve γ is said to be of class C^k iff M is a C^k-manifold and iff there is a C^k-curve $\gamma'\colon (a',b') \to M$ with $I \subseteq (a',b')$ and $\gamma'|_I = \gamma$. By a path, we will understand a curve which is C^∞ or analytic (C^ω) and defined on some closed interval.

[1] Here $C_0(\mathbb{R})$ denotes the set of continuous functions on $\mathbb{R}$ that vanish at infinity.