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Riemannian metrics on an infinite dimensional symplectic group

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ABSTRACT

The aim of this paper is the geometric study of the symplectic operators which are a perturbation of the identity by a Hilbert–Schmidt operator. This subgroup of the symplectic group was introduced in Pierre de la Harpe's classical book of Banach–Lie groups. Throughout this paper we will endow the tangent spaces with different Riemannian metrics. We will use the minimal curves of the unitary group and the positive invertible operators to compare the length of the geodesic curves in each case. Moreover we will study the completeness of the symplectic group with the geodesic distance.

1. Introduction

The symplectic group has many applications in quantum theory with infinitely many degrees of freedom, i.e. in canonical quantum field theory, string theory, statistical quantum physics and solution theory. According to Shale's definitions [17], if we have a quantization $R(.)$ of the real Hilbert space $\Sigma(\mathcal{H})$ it is of interest to determine the subgroups of the symplectic group consisting of those g for which exists a unitary transformation $Y(g)$ such that $R(gz) = Y(g)R(z)Y(g)^{-1}$ for all $z \in \mathcal{H}$. Let $|g| := (g^*g)^{1/2}$ be the absolute value operator, in [17] it was proved that in the case of Fock–Cook quantization (see [5] for some background) the subgroup is $\{g : |g| - 1 \text{ is Hilbert–Schmidt}\}$.

In this paper we study a variant of this subgroup, in which g is only a perturbation of the identity by a Hilbert–Schmidt operator. In classical finite dimensional Riemannian theory it is well known the fact that given two points there is a minimal geodesic curve that joins them and this is equivalent to the completeness of the metric space with the geodesic distance; this is the Hopf–Rinow theorem. In the infinite dimensional case this is no longer true. In [15] and [3], McAlpin and Atkin showed in two examples how this theorem can fail. The main result of this paper establishes that if we consider the left invariant metric in the restricted

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symplectic group then its geodesic distance makes of the group a complete metric space. In the process to do it, we use the existence of a smooth polar decomposition in the group; this will allow us to define a mixed metric related to the unitary and positive part of the group. In this way we will use minimality results of the restricted unitary group $U_2(\mathcal{H})$ (see [1]) and we also prove some geometric properties of the symplectic positive operators with different Riemannian metrics.

2. Background and definitions

Let $\mathcal{H}$ be an infinite dimensional real Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the space of bounded operators. Denote by $\mathcal{B}_2(\mathcal{H})$ the Hilbert–Schmidt class

$$\mathcal{B}_2(\mathcal{H}) = \{a \in \mathcal{B}(\mathcal{H}) : Tr(a^*a) < \infty\}$$

where Tr is the usual trace in $\mathcal{B}(\mathcal{H})$. This space is a Hilbert space with the inner product

$$< a, b >= Tr(b^*a).$$

The norm induced by this inner product is called the 2-norm and denoted by

$$\|a\|_2 = Tr(a^*a)^{1/2}.$$

The usual operator norm will be denoted by $\| \ \|$.

If $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is any subset of operators we use the subscript h (resp. ah) to denote the subset of Hermitian (resp. anti-Hermitian) operators of it, i.e. $\mathcal{A}_h = \{x \in \mathcal{A} : x^* = x\}$ and $\mathcal{A}_{ah} = \{x \in \mathcal{A} : x^* = -x\}$.

We fix a complex structure; that is a linear isometry $J \in \mathcal{B}(\mathcal{H})$ such that,

$$J^2 = -1 \text{ and } J^* = -J.$$

The symplectic form w is given by $w(\xi, \eta) = \langle J\xi, \eta\rangle$.

We denote by $GL(\mathcal{H})$ the group of invertible operators, with $GL(\mathcal{H})^+$ the space of positive invertible operators, and by $\mathrm{Sp}(\mathcal{H})$ the subgroup of invertible operators which preserve the symplectic form, that is $g \in \mathrm{Sp}(\mathcal{H})$ if $w(g\xi, g\eta) = w(\xi, \eta)$. Algebraically

$$\mathrm{Sp}(\mathcal{H}) = \{g \in GL(\mathcal{H}) : g^*Jg = J\}.$$

This group is a Banach–Lie group and its Banach–Lie algebra is given by

$$\mathfrak{sp}(\mathcal{H}) = \{x \in \mathcal{B}(\mathcal{H}) : xJ = -Jx^*\}.$$

Denote by $\mathcal{H}_J$ the Hilbert space $\mathcal{H}$ with the action of the complex field $\mathbb{C}$ given by J, that is; if $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}$ and $\xi \in \mathcal{H}$ we can define the action as $\lambda\xi := \lambda_1\xi + \lambda_2 J\xi$ and the complex inner product as $< \xi, \eta >_{\mathbb{C}} =< \xi, \eta > -iw(\xi, \eta)$.

Denote by $\mathcal{B}(\mathcal{H}_J)$ the space of bounded complex linear operators in $\mathcal{H}_J$. A straightforward computation shows that $\mathcal{B}(\mathcal{H}_J)$ consists of the elements of $\mathcal{B}(\mathcal{H})$ which commute with J.

One property that we will use in this paper is the stability of the adjoint operation. We give a short proof of this fact.

Proposition 2.1. *If $g \in \mathrm{Sp}(\mathcal{H})$ then $g^* \in \mathrm{Sp}(\mathcal{H})$.*