J. Math. Anal. Appl. 428 (2015) 1085–1116

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Entire solutions with asymptotic self-similarity for elliptic equations with exponential nonlinearity ☆

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ARTICLE INFO

Article history:
Received 15 October 2014
Available online 27 March 2015
Submitted by P.J. McKenna

Keywords:
Semilinear elliptic equation
Exponential nonlinearity
Entire solution
Singular solution
Asymptotic self-similarity
Dirichlet problem

ABSTRACT

We consider the elliptic equation $\Delta u + K(|x|)e^u = 0$ in $\mathbf{R}^n \backslash \{0\}$ with $n > 2$ when for $\ell > -2$, $K(r)$ behaves like r^ℓ near 0 or ∞. The asymptotic behavior of radial solutions at ∞ is described by $-(2+\ell)\log r$ for $\ell > -2$ and $-\log\log r$ for $\ell = -2$. When $r^{-\ell}K(r) \to c > 0$ as $r \to \infty$ and $r \to 0$, regular radial solutions at ∞ and singular radial solutions at 0 exhibit self-similarity at ∞ and 0, respectively. Singular solutions with the asymptotic self-similarity exist uniquely in the radial class. Moreover, for $n \geq 10 + 4\ell$, separation of any two radial solutions with the asymptotic self-similarity may happen, while intersection of two solutions may occur for $2 < n < 10 + 4\ell$. In particular, for $n \geq 10 + 4\ell$ with $\ell > -2$, if $K\ (\not\equiv 0)$ satisfies that $r^2 K(r) \to 0$ as $r \to 0$ and $0 \leq k(r) = r^{-\ell}K(r) \leq \frac{n-2}{4(2+\ell)} \inf_{0<s\leq r} k(s)$ for $r > 0$, then any two radial solutions do not intersect each other and each radial solution is linearly stable. When $n \geq 10 + 4\ell$, we apply the global results to prove the uniqueness of positive radial solutions for the Dirichlet problem with zero data on a ball.

1. Introduction

In this paper, we study the elliptic equation

$$\Delta u + K(|x|)e^u = 0, \tag{1.1}$$

where $n > 2$, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, and K is a continuous function in $\mathbf{R}^n \backslash \{0\}$. When $n = 2$, the equation is related to Gaussian curvature equation in conformal geometry. The equation with dimension 3 appears in solid fuel ignition model and the Vlasov–Poisson–Boltzmann system [4]. Moreover, in dimension higher than 2, the equation is compared with the supercritical Lane–Emden equation

$$\Delta u + K(|x|)u^p = 0 \tag{1.2}$$

☆ This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2010-0012580).

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http://dx.doi.org/10.1016/j.jmaa.2015.03.036
0022-247X/

with p bigger than the critical exponent $\frac{n+2+2\ell}{n-2}$ where K in (1.1) and (1.2) behaves like $|x|^\ell$ for some $\ell > -2$. Each equation contains the scale invariance due to nonlinearity, which leads to the asymptotic self-similarity of solutions. Recent work in [2] investigated the role of the asymptotic self-similarity of radial solutions for (1.2). The purpose of the paper is to highlight the same perspective in dealing with (1.1) and (1.2), and to characterize the regular solutions and singular solutions of (1.1) in terms of the asymptotic behavior. Radial solutions of (1.1) satisfy the equation

$$u_{rr} + \frac{n-1}{r}u_r + K(r)e^u = 0 \tag{1.3}$$

where $r = |x|$. Under the following condition:

$$\text{(K)} \quad \begin{cases} K(r) \text{ is continuous on } (0,\infty), \\ K(r) \geq 0 \text{ and } K(r) \not\equiv 0 \text{ on } (0,\infty), \\ \int_0 rK(r)\,dr < \infty, \end{cases}$$

(1.3) with $u(0) = \alpha \in \mathbf{R}$ has a unique solution $u \in C^2(0,\varepsilon) \cap C[0,\varepsilon)$ for small $\varepsilon > 0$. By $u_\alpha(r)$ we denote the unique local solution with $u_\alpha(0) = \alpha$. A typical example is the equation

$$u_{rr} + \frac{n-1}{r}u_r + cr^\ell e^u = 0, \tag{1.4}$$

where $c > 0$ and $\ell > -2$. We denote the solution by $\bar{u}_\alpha$ with $\bar{u}_\alpha(0) = \alpha$. The scale invariance of (1.4) is explained by

$$\bar{u}_\alpha(r) = \alpha + \bar{u}_0(e^{\frac{\alpha}{2+\ell}}r), \tag{1.5}$$

and the invariant singular solution is given by

$$\bar{U}_c(r) := -(2+\ell)\log r + \log(2+\ell)(n-2) - \log c.$$

We call this behavior self-similarity. In fact, for every α, $\bar{u}_\alpha(r) = \bar{U}_c(r) + o(1)$ at ∞. For more general equation (1.3), we look for entire solutions u_α satisfying

$$\liminf_{r\to\infty}[u_\alpha(r) + (2+\ell)\log r] > -\infty. \tag{1.6}$$

The first objective of the paper is to verify the existence of entire solutions with the asymptotic self-similarity.

Theorem 1.1. *Let $n > 2$ and $\ell > -2$. Assume that K satisfies* (K) *and $r^{-\ell}K(r)$ is non-increasing in $(0,\infty)$. For every $\alpha \in \mathbf{R}$, (1.3) has an entire solution u_α with (1.6).*

When $r^{-\ell}K(r)$ converges to a positive constant at ∞, $u_\alpha(r)$ is asymptotically self-similar.

Theorem 1.2. *Let $n > 2$ and $\ell > -2$. Assume that K satisfies* (K) *and $r^{-\ell}K(r) \to c$ as $r \to \infty$ for some $c > 0$. Then, every solution u of (1.3) near ∞ satisfies*

$$\lim_{r\to\infty}[u(r) - \log\frac{(2+\ell)(n-2)}{cr^{2+\ell}}] = 0. \tag{1.7}$$

The asymptotic behavior for $\ell = -2$ is described by $-\log\log r$.