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# Turnpike theorem for terminal functionals in infinite horizon optimal control problems

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ABSTRACT

An optimal control problem for continuous time systems described by a special class of multi-valued mappings and quasi-concave utility functions is considered. The objective is defined as an analogue of the terminal functional defined over an infinite time horizon. An upper bound of this functional over all solutions to the system is established. The turnpike property is proved which states that all optimal solutions converge to some unique optimal stationary point.



## 1. Introduction

In this paper the turnpike property is investigated for a special class of non-convex optimal control problems in continuous time. Simply put this property states that, regardless of initial conditions, all *optimal* trajectories spend most of the time within a small neighborhood of some optimal stationary point when the planning period is long enough. For a classification of different definitions of the turnpike property, we refer the reader to [1,5,13,16,24], and also [2] for the so called *exponential* turnpike property. Possible applications in Markov Games can be found in a recent study [11].

Many approaches have been developed when considering continuous time and discrete time systems. The type of functional involved turns out to be very crucial in the proof of the turnpike property. Discounted and undiscounted integrals are the most commonly studied functionals. Among the most successful approaches developed for these types of functionals, we mention the approaches developed by Rockafellar [21,22] and by Scheinkman, Brock and collaborators (see, for example, [12]). Several other approaches in this area have been developed including those considering special classes of problems (e.g. [10,17,23,25]). An interesting class of control problems considered in [8,9] involves long run average cost functions where the asymptotic behavior of optimal solutions is defined in terms of a probability measure.

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This paper considers a special class of terminal functionals defined as a lower limit at infinity of utility functions. This approach is introduced in [14] where stability results are established for some classes of non-convex problems with applications to environment pollution models. This class of terminal functionals is also used to establish the turnpike theory in terms of statistical convergence [15,18] and $A$-statistical convergence [4], where the convergence of optimal trajectories to some stationary point is proved in the sense of "weak" convergence while ordinary convergence may not be true.

In this paper the turnpike property is established for optimal control problems involving continuous time systems described by differential inclusions. It generalizes some results from [14] obtained for a particular macroeconomic model of air pollution and establishes the turnpike property for a much broader class of optimal control problems by relaxing the assumptions imposed on the set of stationary points as well as on the utility function.

In this study the set of stationary points is not assumed to be bounded as required in the proof of the turnpike property in [14]. Moreover, the utility function is assumed to be quasi concave (instead of concavity in [14]). Obviously, a concave function is also quasi concave but not vice versa; for example, any monotonically increasing or decreasing function is quasi concave. Note that utility functions are often used to describe preferences that are usually assumed to be convex. If a preference relation is given by a continuous utility function, then this preference is convex if and only if the utility function is quasi concave. In this sense, the class of quasi concave utility functions is in some meaningful sense the largest class of functions representing convex preferences.

The assumptions and techniques used in this paper are essentially different from those developed for discrete systems in [4,15,18] where the main assumptions involve both the multi-valued mapping and the utility function. The main assumptions of this paper are imposed on the multi-valued mapping trying to keep (as much as possible) the utility function arbitrary. In this way we establish the class of multi-valued mappings (called class $\mathcal{A}$) for which the turnpike property is true for any quasi concave utility function.

The reminder of the article is organized as follows. In the next section we formulate the problem and provide the notations and assumptions used throughout the paper. Section 3 presents the main results of the paper demonstrated with examples. Some preliminary results are provided in Section 4. The main theorems are proved in Section 5.

## 2. Problem formulation and assumptions

Consider the system

$$\dot{x}(t) \in a(x(t)), \qquad \text{a.e.} \;\; t \geq 0; \tag{1}$$

where $x$ is an element of the Euclidean space $R^n$. The multi-valued mapping $a$ is defined on a convex closed set $\mathcal{D}_a$ with non-empty interior, has compact images and is upper semi-continuous (u.s.c.) in the Hausdorff metric. The assumption that $\mathcal{D}_a$ has a non-empty interior is not restrictive; otherwise one could consider system (1) in a subspace of $R^n$ (the affine hull of $\mathcal{D}_a$) by reducing the dimensionality of the space where the corresponding multi-valued mapping has a non-empty interior.

We will use the notation $a(A) = \cup_{x \in A} a(x)$ and, given a point $x$, we do not distinguish between $a(x)$ and $a(\{x\})$. Throughout the paper, " $\cdot$ ", "co" and "int" stand for the scalar product, convex hull and interior, respectively.

### *2.1. Solutions to (1)*

An absolutely continuous function $\mathbf{x} = x(t)$, $t \geq 0$, satisfying (1) is called a solution. We assume that system (1) has a bounded solution defined on an infinite horizon $[0, \infty)$. This is not a restrictive assumption