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Quantitative results for Halpern iterations of nonexpansive mappings

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ABSTRACT

We give a rate of metastability for Halpern's iteration relative to a rate of metastability for the resolvent for nonexpansive mappings in uniformly smooth Banach spaces, extracted from a proof due to Xu. In Hilbert space, the latter is known, so we get an explicit rate of metastability. We also extract a rate of asymptotic regularity for general normed spaces. Such rates have already been extracted by Kohlenbach and Leuştean for different and incomparable conditions on (λ_n). The proof analyzed in this paper is also more effective than the proofs treated by Kohlenbach and Leuştean in that it does not use Banach limits or weak compactness, which makes the extraction particularly efficient. Moreover, we also give an equivalent axiomatization of uniformly smooth Banach spaces. This paper is part of an ongoing case study of proof mining in nonlinear fixed point theory.

1. Introduction

Let H be a Hilbert space, $C \subseteq H$ be a convex closed subset and $S : C \to C$ be nonexpansive. For some starting point $x_0 \in C$, some anchor $u \in C$ and some sequence $(\alpha_n) \subset [0, 1]$, the Halpern iteration is defined by

$$x_{n+1} := \alpha_n u + (1 - \alpha_n) S x_n. \tag{1.1}$$

The scheme was first introduced in [4], albeit only when C is the closed unit ball and $u = 0$. For this case, Halpern [4] also gave a set of necessary and a set of sufficient conditions for (α_n) under which the scheme (1.1) converges strongly to a fixed point of S. However, Halpern's conditions allowed no conclusion whether the natural choice $\alpha_n = 1/(n+1)$ is admitted. Wittmann [21] answered this question in the affirmative in 1992: If S has a fixed point and the sequence (α_n) satisfies

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$$\text{(i)} \quad \lim_{n\to\infty} \alpha_n = 0, \qquad \text{(ii)} \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \qquad \text{(iii)} \quad \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

then the Halpern iteration converges strongly to the fixed point closest to the starting point x_0.

Using proof-theoretic methods exhibited in Kohlenbach [5,9], Leuştean [13] extracted from Wittmann's proof a rate of asymptotic regularity for general normed and even hyperbolic spaces, i.e., a rate of convergence for $\|Tx_n - x_n\| \to 0$ under the assumption that the Halpern iteration remains bounded, which is always the case if S has a fixed point. The rate is highly uniform in the sense that it does not depend on the set C, the operator S or the specific choice of the sequence (α_n), but only on witnesses for the existential quantifiers in conditions (i) to (iii) above and a bound on, in essence, the sequence (x_n).

Strong convergence is then established by Wittmann using the metric projection of x_0 onto the fixed point set and weak sequential compactness applied to the iteration sequence. As shown by Avigad, Gerhardy and Towsner in [1], there cannot be a computable bound on the rate of convergence even for the special case where $\alpha_n = 1/n + 1$ and S is linear. In this case, the Halpern iteration coincides with the ergodic average, and so Wittmann's theorem implies von Neumann's mean ergodic theorem.

On the other hand, a uniform rate of metastability (in the sense of Tao [20,19]) is guaranteed to exist by a general metatheorem of Kohlenbach [6] and was extracted by the same author in [7]. A rate of metastability is a bound on the existential quantifier in the Herbrand normal form of the statement that (x_n) is Cauchy:

$$\forall \varepsilon > 0 \forall g : \mathbb{N} \to \mathbb{N} \exists n \leq \Phi(\varepsilon, g) \forall i, j \in [n; n + g(n)] \Big(\|x_i - x_j\| < \varepsilon \Big), \tag{1.2}$$

where $[n; n + g(n)] := \{n, n + 1, \ldots, n + g(n)\}$. The bound is highly uniform in the sense that it does not depend on the operator S, the starting point x_0, the anchor u or the specific Hilbert space. Apart from rates of convergence and divergence for the conditions (i) to (iii), it only depends on an upper bound on the distance of the starting point from some fixed point of S. In the case $\alpha_n = 1/n + 1$, Kohlenbach [7] also improved the exponential rate of asymptotic regularity to a quadratic one. Moreover, these results were also generalised to CAT(0) spaces [8] and CAT(κ) spaces [14].

Closely related to Wittmann's result is the following

Theorem 1.1. *(See Browder [2].) Let H be a Hilbert space, S a nonexpansive mapping of H into H. Suppose that there exists a bounded closed convex subset C of H mapped by S into itself. Let u be an arbitrary point of C, and for each t with $0 < t < 1$, let $S_t x = tu + (1 - t)Sx$.*

Then S_t is a strict contraction of H with ratio t, S_t has a unique fixed point z_t in C, and z_t converges as $t \to 0$ strongly in H to a fixed point v of S in C. The fixed point v is uniquely specified as the fixed point of S closest to u.

The proof is structured similarly to the proof of Wittmann's theorem in that its ineffective part consists of a projection onto the fixed point set and weak sequential compactness, this time applied to (z_t). In fact, the proof theoretic analysis of Browder's theorem, also carried out in [7], exhibits interesting parallels to the aforementioned one.

There is also an elementary proof due to Halpern [4] for the special case where C is the closed unit ball of H, which can easily be generalised to arbitrary bounded closed convex subsets. The ineffectivity of Halpern's proof stems from the monotone convergence principle, i.e., that every monotone sequence in the real unit interval converges. A metastable version of this can be found on page 30 of [5]. Using this, a simpler rate of metastability was extracted in [7].