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Gevrey regularity for integro-differential operators

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ABSTRACT

We prove for some singular kernels $K(x, y)$ that viscosity solutions of the integro-differential equation

$$\int_{\mathbb{R}^n} [u(x + y) + u(x - y) - 2u(x)]\ K(x, y)dy = f(x)$$

locally belong to some Gevrey class if so does f. The fractional Laplacian equation is included in this framework as a special case.

1. Introduction

Recently, a great attention has been devoted to equations driven by nonlocal operators of fractional type. From the physical point of view, these equations take into account long-range particle interactions with a power-law decay. When the decay at infinity is sufficiently weak, the long-range phenomena may prevail and the nonlocal effects persist even on large scales (see e.g. [7,19,21]).

The probabilistic counterpart of these fractional equations is that the underlying diffusion is driven by a stochastic process with power-law tail probability distribution (the so-called Pareto or Lévy distribution), see for instance [28,26]. Since long relocations are allowed by the process, the diffusion obtained is sometimes referred to with the name of anomalous (in contrast with the classical one coming from Poisson distributions). Physical realizations of these models occur in different fields, such as fluid dynamics (and especially quasi-geostrophic and water wave equations), dynamical systems, elasticity and micelles, see, among the

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others [25,9,10,22]. Also, the scale invariance of the nonlocal probability distribution may combine with the intermittency and renormalization properties of other nonlinear dynamics and produce complex patterns with fractional features. For instance, there are indications that the distribution of food on the ocean surface has scale invariant properties (see e.g. [27] and references therein) and it is possible that optimal searches of predators reflect these patterns in the effort of locating abundant food in sparse environments, also considering that power-law distribution of movements allows the individuals to visit more sites than the classical Brownian situation (see e.g. [2,14]).

The regularity theory of integro-differential equations has been extensively studied in continuous and smooth spaces, see e.g. [23,5,1,11]. The purpose of this paper is to deal with the regularity theory in a Gevrey framework. The proof combines a quantitative bootstrap argument developed in [1] and the classical iteration scheme of [18,17]. Here the bootstrap argument is more delicate than in the classical case due to the nonlocality of the operator, since the value of the function in a small ball is affected by the values of the function everywhere, not only in a slightly bigger ball; in particular the derivatives of the function cannot be controlled in the whole space and a suitable truncation argument is needed.

Before stating the main results of the paper, we recall the definition of Gevrey function. For a detailed treatment of the theory of Gevrey functions and their relation with analytic functions we refer to [16,20]. Let $\Omega \in \mathbb{R}^n$ be an open set, we define for any fixed real number $\sigma \geq 1$ the class $\mathcal{G}^\sigma(\Omega)$ of Gevrey functions of order σ in Ω. This is the set of functions $f \in \mathcal{C}^\infty(\Omega)$ such that for every compact subset Θ of Ω there exist positive constants V and Γ such that for all $i \in \mathbb{N}$

$$\left\|D^i f\right\|_{\mathrm{L}^\infty(\Theta)} \leq V\,\Gamma^i\,(i!)^\sigma\,.$$

We remark that the spaces $\mathcal{G}^\sigma(\Omega)$ form a nested family, in the sense that $\mathcal{G}^\sigma(\Omega) \subseteq \mathcal{G}^\tau(\Omega)$ whenever $\sigma \leq \tau$ and furthermore the inclusion is strict whenever the inequality is. Clearly the class $\mathcal{G}^1(\Omega)$ coincides with $\mathcal{C}^\omega(\Omega)$, that of analytical functions. It should be stressed that both inclusions

$$\mathcal{C}^\omega(\Omega) \subset \bigcap_{\sigma>1} \mathcal{G}^\sigma(\Omega) \qquad\qquad \bigcup_{\sigma\geq 1} \mathcal{G}^\sigma(\Omega) \subset \mathcal{C}^\infty(\Omega)$$

are strict, see [20]. The notion of Gevrey class of functions is quite useful in applications. For instance, it possess a nice characterization in Fourier spaces. Moreover, cut-off functions are never analytic, but they may be chosen to belong to a Gevrey space. Roughly speaking, for a smooth function f the notion of Gevrey order measures "how much" the Taylor series of f diverges.

As in [1] we consider a quite general kernel $K = K(x,y) : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to (0,+\infty)$ satisfying some structural assumptions. From now we assume that $s \in (1/2, 1)$.

We suppose that K is close to the kernel of the fractional Laplacian in the sense that

$$\begin{cases} \text{there exist } a_0, r_0 \text{ and } \eta \in (0, a_0/4) \text{ such that} \\ \left|\dfrac{|y|^{n+2s}K(x,y)}{2-2s} - a_0\right| \leq \eta \qquad \text{for all } x \in B_1,\ y \in B_{r_0}\setminus\{0\}\,. \end{cases} \tag{1.1}$$

Since we are interested in the Gevrey regularity, in order to ensure that our solutions are $\mathcal{C}^\infty$ we assume that $K \in \mathcal{C}^\infty\left(B_1 \times (\mathbb{R}^n \setminus \{0\})\right)$ and moreover

$$\begin{cases} \text{for all } k \in \mathbb{N}\cup\{0\} \text{ there exist } \mathrm{H}_k > 0 \text{ such that} \\ \left\|D_x^\mu D_y^\theta K(\cdot,y)\right\|_{\mathrm{L}^\infty(B_1)} \leq \dfrac{\mathrm{H}_k}{|y|^{n+2s+|\theta|}} \qquad \text{for all } \mu,\theta \in \mathbb{N}^n,\ |\mu|+|\theta| = k,\ y \in B_{r_0}\setminus\{0\}\,. \end{cases} \tag{1.2}$$

Furthermore, since we need a quantitative asymptotic control on the tails of the derivatives of K, we assume that