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Bifurcation for a logistic elliptic equation with nonlinear boundary conditions: A limiting case

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ABSTRACT

We investigate bifurcation from the zero solution for a logistic elliptic equation with a sign-definite nonlinear boundary condition. In view of the lack of regularity of the term on the boundary, the abstract theory on bifurcation from simple eigenvalues due to Crandall and Rabinowitz does not apply. A regularization procedure and a topological method due to Whyburn are used to prove the existence and the global behavior at infinity of a subcontinuum of nontrivial non-negative weak solutions. The direction of the bifurcation component at zero is also investigated. This paper treats a limiting case of our previous work [19], where the case of sign-changing nonlinear boundary conditions is considered.

1. Introduction and statements of main results

Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$. We consider the following nonlinear elliptic problem.

$$\begin{cases} -\Delta u = \lambda(m(x)u - a(x)|u|^{p-2}u) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda b(x)|u|^{q-2}u & \text{on} \quad \partial\Omega, \end{cases} \tag{P_λ}$$

where

- $\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$ is the usual Laplacian in $\mathbb{R}^N$,
- $\lambda > 0$,
- $1 < q < 2 < p < \infty$, and $p \leq 2^* = \frac{2N}{N-2}$ if $N > 2$,
- $m, a \in \mathcal{C}^\alpha(\overline{\Omega})$, $\alpha \in (0,1)$, $m^+ := \max(m, 0) \not\equiv 0$, and $a > 0$ in $\overline{\Omega}$,

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- $b \in \mathcal{C}^{1+\alpha}(\partial\Omega)$,
- $\mathbf{n}$ is the unit outer normal to the boundary $\partial\Omega$.

A function $u \in H^1(\Omega)$ is said to be a *weak solution* of (P_λ) if it satisfies

$$\int_\Omega \nabla u \nabla w - \lambda \int_\Omega muw + \lambda \int_\Omega a|u|^{p-2}uw - \lambda \int_{\partial\Omega} b|u|^{q-2}uw = 0, \quad \forall w \in H^1(\Omega).$$

If, in addition, $u \geq 0$ and $u \not\equiv 0$ then we say that u is a *nontrivial non-negative* weak solution of (P_λ). By a bootstrap argument for nonlinear boundary conditions (see Rossi [21]), we see that any weak solution belongs to $\mathcal{C}^2(\Omega) \cap \mathcal{C}^\theta(\overline{\Omega})$ for some $\theta \in (0,1)$. From the strong maximum principle, we deduce that nontrivial non-negative weak solutions of (P_λ) are strictly positive in Ω. However, it seems difficult to deduce their positivity on the whole of $\overline{\Omega}$, since the boundary point lemma is not applicable. At least we know that the set $\{x \in \partial\Omega : u(x) = 0\}$ has no interior points in the relative topology of $\partial\Omega$, and it is contained in the set $\{x \in \partial\Omega : b(x) \leq 0\}$, see [20, Proposition 5.1]. By a *positive classical solution* of (P_λ) we mean a function in $\mathcal{C}^{2+\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$ which satisfies (P_λ) in the classical sense and is positive in $\overline{\Omega}$. If u is a nontrivial non-negative weak solution of (P_λ) then we can see that $u > 0$ on $\partial\Omega$ if and only if $u \in \mathcal{C}^1(\overline{\Omega})$, and in this case, u is a positive classical solution.

The equation $-(1/\lambda)\Delta u = mu - a|u|^{p-2}u$ in Ω arises from population dynamics [9]. The unknown function u stands for the population density of some species inhabiting the region Ω, and $1/\lambda$ is the diffusion coefficient of this species. Our nonlinear boundary condition suggests that the flux rate $(1/\lambda)\nabla u \cdot \mathbf{n}$ of the population on $\partial\Omega$ is incoming or outgoing (according to the sign of $b(x)$) and depends nonlinearly on u as $|u|^{q-2}u$. Logistic type equations with Neumann boundary conditions and powerlike nonlinearities, such as (P_λ), have been studied extensively in the last decade. We refer to Umezu [23,24,26–28] for a bifurcation analysis in the case $q > 2$, to Morales-Rodrigo and Suárez [16] and García-Melián, Morales-Rodrigo, Rossi, and Suárez [13] for a detailed analysis of existence, uniqueness, multiplicity, and asymptotic behavior of positive solutions (in the case of positive constant coefficients), to Morales-Rodrigo and Suárez [17] for uniqueness of positive solutions, and to Cano-Casanova [7,8] for nonlinear mixed boundary conditions when a is non-negative and $q > 2$. We also refer to our previous works [19,20], where we deal with the case $q < 2$. In [19], the case where a is non-negative and b is indefinite is studied by a combination of variational and bifurcation techniques, which is extended to the case where a changes sign in [20].

If $b \geq 0$ and $b \not\equiv 0$ then the positive classical solutions set of (P_λ) is rather simple. Indeed, by means of the super and subsolutions method, we can prove the following existence and uniqueness result:

Theorem 1.1. *Let $b \geq 0$ and $b \not\equiv 0$. Then for any $\lambda > 0$ there exists a positive classical solution of (P_λ), which is unique among nontrivial non-negative weak solutions of (P_λ). Moreover, this solution converges to c_0 in $\mathcal{C}^{2+\alpha}(\overline{\Omega})$ as $\lambda \to 0^+$, where c_0 is the unique positive zero of the function $t \mapsto t^{2-q}\int_\Omega m - t^{p-q}\int_\Omega a + \int_{\partial\Omega} b$.*

Remark 1.2.

(1) Theorem 1.1 holds for an arbitrary $m \in \mathcal{C}^\alpha(\overline{\Omega})$ without the assumption $m^+ \not\equiv 0$.
(2) By Green's formula, positive classical solutions u_λ of (P_λ) satisfy

$$\int_\Omega \left(mu_\lambda - au_\lambda^{p-1}\right) + \int_{\partial\Omega} bu_\lambda^{q-1} = 0.$$

Hence, we can see that if $c > 0$ is a constant and $u_\lambda \to c$ in $\mathcal{C}(\overline{\Omega})$ as $\lambda \to 0^+$ then $c^{2-q}\int_\Omega m - c^{p-q}\int_\Omega a + \int_{\partial\Omega} b = 0$.