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# A Hilbert-type integral inequality in the whole plane related to the hypergeometric function and the beta function ☆

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ABSTRACT

A new Hilbert-type integral inequality in the whole plane with the non-homogeneous kernel and parameters is given. The constant factor related to the hypergeometric function and the beta function is proved to be the best possible. As applications, equivalent forms, the reverses, some particular examples, two kinds of Hardy-type inequalities, and operator expressions are considered.



## 1. Introduction

If $f(x), g(y) \geq 0$ satisfy

$$0 < \int_0^\infty f^2(x)dx < \infty$$

and

$$0 < \int_0^\infty g^2(y)dy < \infty,$$

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then we have the following Hilbert's integral inequality (cf. [2]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \left( \int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy \right)^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor $\pi$ is the best possible. The inequality (1.1) is very important in Mathematical Analysis and its applications (cf. [2,10]). In recent years, by the use of the method of weight functions, a number of extensions of (1.1) were given by Yang (cf. [25]). Noticing that inequality (1.1) is a homogeneous kernel of degree $-1$, in 2009, a survey of the study of Hilbert-type inequalities with the homogeneous kernels of degree equal to negative numbers and some parameters is given in [26]. Recently, some inequalities with the homogeneous kernels of degree 0 and non-homogeneous kernels have been proved (cf. [1,17,19–21,27]). Other kinds of Hilbert-type inequalities are shown in [8,9,11–14,23]. All of the above integral inequalities are constructed in the quarter plane of the first quadrant.

In 2007, Yang [22] presented a new Hilbert-type integral inequality in the whole plane, as follows:

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{(1+e^{x+y})^\lambda} dxdy < B(\frac{\lambda}{2}, \frac{\lambda}{2})( \int_{-\infty}^\infty e^{-\lambda x} f^2(x)dx \int_{-\infty}^\infty e^{-\lambda y} g^2(y)dy)^{\frac{1}{2}}, \tag{1.2}$$

where the constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ $(\lambda > 0)$ is the best possible.

If $0 < \lambda < 1$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, Yang [24] derived another new Hilbert-type integral inequality in the whole plane. Namely, he proved that

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{|1+xy|^\lambda} f(x)g(y)dxdy < k_\lambda \left[ \int_{-\infty}^\infty |x|^{p(1-\frac{\lambda}{2})-1} f^p(x)dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^\infty |y|^{q(1-\frac{\lambda}{2})-1} g^q(y)dy \right]^{\frac{1}{q}}, \tag{1.3}$$

where the constant factor

$$k_\lambda = B(\frac{\lambda}{2}, \frac{\lambda}{2}) + 2B(1-\lambda, \frac{\lambda}{2})$$

is still the best possible. Furthermore, Yang et al. [3–5,16,18,24,28–31] proved as well some new Hilbert-type integral inequalities in the whole plane.

In this paper, using methods from Real Analysis and by estimating the weight functions, a new Hilbert-type integral inequality in the whole plane with the non-homogeneous kernel and multi-parameters is shown, which gives an extension of (1.3). The constant factor related to the hypergeometric function and the beta function is proved to be the best possible. As applications, equivalent forms, the reverses, some particular examples, two kinds of Hardy-type inequalities, and operator expressions are considered.

## 2. Some lemmas

Initially, we introduce the following formula of the hypergeometric function $F$ (cf. [15]): If $\mathrm{Re}(\gamma) > \mathrm{Re}(\theta) > 0$, $|\arg(1-z)| < \pi$, $(1-zt)^{-\alpha}|_{z=0} = 1$, then

$$F(\alpha, \theta, \gamma, z) := \frac{\Gamma(\gamma)}{\Gamma(\theta)\Gamma(\gamma-\theta)} \int_0^1 t^{\theta-1}(1-t)^{\gamma-\theta-1}(1-zt)^{-\alpha} dt,$$

where