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Stability and uniqueness for an integral geometry problem with a weight function

Zekeriya Ustaoglu

Department of Mathematics, Faculty of Arts and Sciences, Bulent Ecevit University, 67100, Zonguldak, Turkey

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ABSTRACT

We consider the integral geometry problems including a weight function along the family of plane curves of given curvatures and prove a stability estimate and the uniqueness of the solution of these problems on the space of sufficiently smooth functions. Then, these stability and uniqueness results are investigated for the problems along the families of some special plane curves and some consequences are presented.

1. Introduction

The problems of integral geometry, in the sense that based on the work of Radon [23], consist in determining a function by its given integrals of this function (or the integrals are taken with a certain weight multiplier) over a family of manifolds. In fact, the classical Radon transform maps a function to its integrals over all hyperplanes in n-dimensional space, and in general, the transforms involving integrations over curved surfaces and/or weight are called generalized Radon transforms.

It is worth noting that the term "integral geometry", in a different sense than the one described above, was first used related to the problems on geometric probability and convex bodies (see, e.g., [25]). Moreover, the theory of integral geometry has been considered in various aspects, e.g., related with the theory of Lie group representations, the inverse problems for differential equations (see, e.g., [3,12,24,26] and the references therein) and the practical applications such as tomography (see, e.g., [18,20]). An integral geometry problem over the family of straight lines in the plane modeling the X-rays and applicable to the problems of radiology and radiotherapy was first indicated in [7]. Because of their theoretical importance and many practical applications, a considerable interest has been devoted to other families of curves in the plane as well as the straight lines and with the weight function is not unity as well as unity. In the study of these problems, one

E-mail address: zekeriyaustaoglu@karaelmas.edu.tr.

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of the main matters is to obtain an analytical formula or a numerical reconstruction algorithm expressing the unknown function in terms of its given integrals. Another important question of integral geometry is under which conditions one can uniquely determine the unknown function and derive a meaningful stability estimate.

Concerning the generalized Radon or ray transforms several methods, such as circular harmonic decomposition, Kolmogorov's superposition theorem, A-analytic functions, etc. are utilized on the above mentioned problems on the injectivity and inversion [4,8,10,19,21]. By using analytic microlocal analysis, a generic uniqueness result and a stability estimate of a weighted ray transform are given in [11]. In [6], a famous example of nonuniqueness of generalized Radon transform is presented. [5, Chapters 2 and 3] provides a useful introduction and [9] presents some important studies on the related problems.

On the space of sufficiently smooth functions, the uniqueness theorems and the stability estimates are given for the linear and the nonlinear problems of integral geometry along some regular family of curves by Mukhometov in [16,17]. These results are based on reducing the problem of integral geometry to an equivalent boundary value problem for a second order partial differential equation of mixed type and using the method of energy estimates, and are assumed to be one of the strongest and the most general results on the stability and uniqueness of integral geometry problems on plane curves (see, e.g., [15, Chapter 6, Section 5] or [27, Chapter 1]). In fact, investigating the uniqueness of solution of a problem of integral geometry by reducing it to an equivalent inverse problem for a differential equation was first carried out in [14].

On using some extension of class of unknown functions and with a different choice of the coordinates, some solvability results on the integral geometry problem along a regular family of curves, the assumptions on which do not coincide with the ones of Mukhometov (see Remark 1 in Section 2), are given by Amirov in [2] (see also [3, Chapter 1], [28] and the references therein). There exist families of curves which satisfy the assumptions of Amirov but do not satisfy the assumptions of Mukhometov. So, the motivation of this study is establishing the stability and uniqueness of the solution of the integral geometry problems including weight function on the space of sufficiently smooth functions with the choice of the coordinates and the family of the curves, more general than the ones of Mukhometov, as in [2]. Thus, the results obtained provide a contribution to the theory of integral geometry and generalized Radon transforms with the new stability estimates and the uniqueness results for the problems along some more general family of plane curves.

In the next section preliminary notations and assumptions are provided and a problem of integral geometry (IGP) is stated. In Section 3, a stability estimate and the uniqueness of the solution of IGP are given by Theorem 3 with a general condition on the curvatures of the curves of the family along which the problem is considered. Furthermore, as some consequences of Theorem 3, the stability and uniqueness results for IGP with a less general condition on the curvatures, with the curvatures of a special form, along the family of circles with fixed radius and along the family of circles of varying radius centered on a fixed circle are presented in Corollaries 6, 8, 9 and 10 respectively. The last section is devoted to some conclusions.

2. Preliminaries

Let D be a bounded simply connected domain in $\mathbb{R}^2$ with the boundary $\partial D \in C^1$, which can be represented in the parametric form by

$$x = (\delta_1(t), \delta_2(t)), \quad 0 \le t \le T,$$

where the parameter t is the arc length on ∂D measured from a fixed point on ∂D in a positive direction, and T is the length of ∂D, so

$$(\delta_1(0), \delta_2(0)) = (\delta_1(T), \delta_2(T)).$$