J. Math. Anal. Appl. 428 (2015) 1339–1353

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the second order periodic problem at resonance with impulses

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ARTICLE INFO

Article history:
Received 18 December 2014
Available online 2 April 2015
Submitted by P.J. McKenna

Keywords:
Periodic problem for second order differential equations
Impulses
Resonance problem
Landesman–Lazer condition
Saddle point theorem

ABSTRACT

In this paper we consider the periodic problem for the second order equation at resonance with impulses in the derivative. The impulses are considered at fixed times and depend on the actual value of the solution in a nonlinear way. We formulate rather general sufficient condition in terms of the asymptotic properties of both nonlinear restoring force and nonlinear impulses which generalizes classical Landesman–Lazer condition. Moreover, our condition implies existence results for some open problems with vanishing and oscillating nonlinearities.

1. Introduction

This paper is devoted to the study of the nonlinear second order equations with periodic boundary conditions and with impulses in the derivative at fixed times. The impulses depend on the actual value of the solution in a nonlinear way. The reader is referred to the books [1,14,22] for the general theory of impulsive differential equations. The practical importance of models the solutions of which include instantaneous impulses depending on the position that result in jump discontinuities in velocity, but with no change in position, was stressed in papers [3,4,20]. In our paper we concentrate on the so-called resonance problems and provide sufficient condition for the existence of a solution in terms of the forcing term, restoring force and the impulse functions. Our condition not only generalizes the classical Landesman–Lazer condition but also implies the existence results for problems with vanishing and/or oscillating nonlinearities which remained unsolved so far.

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http://dx.doi.org/10.1016/j.jmaa.2015.03.075

Let us consider the second order periodic problem at resonance with nonlinear impulses at the derivative:

$$\begin{aligned} &x''(t) + m^2 x(t) + f(t, x(t)) = e(t), \quad \text{for a.e. } t \in [0, 2\pi], \\ &x(0) = x(2\pi),\ x'(0) = x'(2\pi), \\ &x(t_j{}^+) = x(t_j{}^-),\ \Delta x'(t_j) := x'(t_j{}^+) - x'(t_j{}^-) = I_j(x(t_j)), \quad j = 1, 2, \dots, p, \end{aligned} \tag{1}$$

where $m, p \in \mathbb{N}$, $f : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $e \in L^1(0, 2\pi)$, $0 < t_1 < \ldots < t_p < 2\pi$ and $I_j : \mathbb{R} \to \mathbb{R}$ is continuous, $j = 1, 2, \dots, p$.

In order to formulate rigorously our condition, we need some notation: Let $H := \{x \in H^1(0, 2\pi) : x(0) = x(2\pi)\}$ be equipped with the scalar product $(x, y) = \int_0^{2\pi} (x'(t)y'(t) + x(t)y(t))\,\mathrm{d}t$ and induced the norm $\|x\| = \left(\int_0^{2\pi} \left[(x'(t))^2 + (x(t))^2\right] \mathrm{d}t\right)^{1/2}$. We will work also with the norm $\|x\|_\infty = \max\limits_{t \in [0,2\pi]} |x(t)|$ on space $C[0, 2\pi]$. We split H as $H = \hat{H} \oplus \bar{H} \oplus \tilde{H}$, where

$$\begin{aligned} \hat{H} &= \mathrm{span}\{1, \sin t, \cos t, \sin 2t, \cos 2t, \dots, \sin(m-1)t, \cos(m-1)t\}, \\ \bar{H} &= \mathrm{span}\{\sin mt, \cos mt\}, \\ \tilde{H} &= \mathrm{span}\{\sin(m+1)t, \cos(m+1)t, \dots\}. \end{aligned}$$

We also split a function $e \in L^1(0, 2\pi)$ as $e = \bar{e} + e^\perp$, where

$$\int_0^{2\pi} e^\perp(t) \sin(mt + \theta)\,\mathrm{d}t = 0 \qquad \text{for all } \theta \in \mathbb{R}.$$

We are now ready to formulate our sufficient condition:

If $(x_n) \subset H$ is a sequence such that $\|x_n\|_\infty \to \infty$ and there exists $\theta_0 \in \mathbb{R}$ such that $\frac{x_n}{\|x_n\|_\infty} \to \sin(mt + \theta_0)$ in $C[0, 2\pi]$, then

$$\lim_{n\to\infty} \left(\int_0^{2\pi} \int_0^{x_n(t)} f(t, s)\,\mathrm{d}s\,\mathrm{d}t - \sum_{j=1}^{p} \int_0^{x_n(t_j)} I_j(s)\,\mathrm{d}s - \int_0^{2\pi} \bar{e}(t) x_n(t)\,\mathrm{d}t \right) = \pm\infty. \tag{SC$_\pm$}$$

We assume that there exists $r \in L^1(0, 2\pi)$ such that

$$|f(t, s)| \le r(t) \tag{f}$$

for a.e. $t \in [0, 2\pi]$ and for all $s \in \mathbb{R}$. We also assume that there exists a constant $c > 0$ such that for all $s \in \mathbb{R}$,

$$|I_j(s)| \le c, \quad j = 1, 2, \dots, p. \tag{I$_\mathrm{j}$}$$

Our main result is then the following theorem.

Theorem 1. *Let us assume (f), $(\mathrm{I_j})$ and let either $(\mathrm{SC})_+$ or $(\mathrm{SC})_-$ hold true. Then problem (1) has at least one solution.*

In the following examples we illustrate connection between our conditions $(\mathrm{SC})_\pm$ and the classical Landesman–Lazer conditions, as well as its various generalizations presented in the literature.