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A linear estimate of the number of limit cycles for some planar piecewise smooth quadratic differential system

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ABSTRACT

In this paper we study a class of planar piecewise smooth quadratic integrable non-Hamiltonian systems, which have a center. By using the averaging method, we give an estimation of the number of limit cycles which bifurcate from the above periodic annulus under the polynomial perturbation of degree n. Our estimation is linear depending on n and it is at least twice the associated estimation of smooth systems.

1. Introduction and statement of the main results

In recent years, stimulated by non-smooth phenomena in the real world, there has been considerable interest in the following planar piecewise smooth differential system

$$(\dot{x}, \dot{y}) = \begin{cases} (P_1(x,y), Q_1(x,y)) & \text{if } S(x,y) > 0, \\ (P_2(x,y), Q_2(x,y)) & \text{if } S(x,y) < 0, \end{cases} \tag{1}$$

which raised from mechanics, electrical engineering and automatic control, see for instance [1]. Here the dot denotes derivative with respect to the variable t. The switching manifold $\Sigma = \{(x,y) : S(x,y) = 0\}$ divides $\mathbb{R}^2$ into two regions, where the systems are smooth in each region.

In the qualitative theory of real planar differential system, one of the important problems is to determine the number of limit cycles. To solve this problem, people have developed many methods. Among them, the bifurcation is a very useful idea. For the bifurcation of smooth systems, people have lots of works, see for instance [12], but for non-smooth systems, as pointed out in [14], almost always the generalization of the bifurcation methods for smooth systems to non-smooth systems is a non-trivial task. Even though many

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results in the classic theory of smooth systems have been shown to be valid for non-smooth systems (see [10]), we have to say that we know little about non-smooth systems.

To produce limit cycles, the first method is the Hopf bifurcation, that is, the limit cycles come from a center or a focus. The non-smooth Hopf bifurcation has been considered by several papers, see for instance [4,5,13,20,24]. In [13,20] the authors show that piecewise smooth linear system can have two limit cycles surrounding the origin. Later on, the authors in [4] prove that piecewise smooth linear systems can have three limit cycles surrounding a unique equilibrium. In the paper [5], the authors obtain that piecewise smooth quadratic system can have nine small amplitude limit cycles. In [24], the authors consider the Hopf bifurcation of piecewise planar Hamiltonian systems.

The second method is the Poincaré bifurcation, that is, the limit cycles come from the periodic annulus. In [9], the authors consider the perturbation of limit cycles for a general planar Filippov system. In [18], the authors obtain the first order Melnikov function for planar piecewise smooth Hamiltonian systems. In [19], the authors study the limit cycles bifurcating from quadratic isochronous systems under discontinuous quadratic polynomial perturbation. In a recent paper [22], the authors consider the number of limit cycles in discontinuous classical Liénard equations.

The third method to generate limit cycles is by perturbing a Homoclinic or a Heteroclinic loop, see for instance [16,17]. In a review paper [2], there are also many other bifurcations which are unique to nonsmooth systems.

However, so far there are few papers in the literature studying the number of limit cycles bifurcating inside the class of piecewise smooth integrable non-Hamiltonian system. In this paper, we consider the following planar piecewise smooth differential system

$$(\dot{x}, \dot{y}) = \begin{cases} (-y(1+ax) + \varepsilon P^+(x,y), x(1+ax) + \varepsilon Q^+(x,y)) & \text{if } x > 0, \\ (-y(1+bx) + \varepsilon P^-(x,y), x(1+bx) + \varepsilon Q^-(x,y)) & \text{if } x < 0, \end{cases} \tag{2}$$

where

$$P^+(x,y) = \sum_{i+j=0}^{n} p_{i,j} x^i y^j, \quad Q^+(x,y) = \sum_{i+j=0}^{n} q_{i,j} x^i y^j,$$
$$P^-(x,y) = \sum_{i+j=0}^{n} s_{i,j} x^i y^j, \quad Q^-(x,y) = \sum_{i+j=0}^{n} t_{i,j} x^i y^j.$$

Since system $(2)_{\varepsilon=0}$ has the first integral $\mu(x,y) = x^2 + y^2$ with respect to $x > 0$ and $x < 0$, the origin is a center by Proposition 2.1 of [6] and system $(2)|_{\varepsilon=0}$ is a piecewise smooth integrable non-Hamiltonian quadratic differential system. It is worth noting that system $(2)|_{\varepsilon=0}$ has an invariant straight line $1+ax=0$ (resp. $1+bx=0$) for the case $a<0$ (resp. $b>0$).

The objective of this paper is to estimate the number of limit cycles which bifurcate from any compact region of the periodic annulus of system (2). Our method is the first order averaging method in [21].

Denote by $H(n)$ the maximum number of limit cycles bifurcating from any compact region of the periodic annulus of system (2) up to the first order averaging method, then we have the following result:

Theorem 1. *Consider system (2) with $|\varepsilon| > 0$ sufficiently small.*

(i) *If $a = b = 0$, then $H(n) = n$.*
(ii) *If $a = 0$, $b \neq 0$ or $a \neq 0$, $b = 0$, then $H(n) = [\frac{n+1}{2}] + n + 1$.*
(iii) *If $ab \neq 0$ and $a = -b$, then $H(n) = [\frac{n+1}{2}] + n$.*
(iv) *If $ab \neq 0$ and $a \neq -b$, then $H(n) = 2[\frac{n+1}{2}] + n + 1$.*