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Elliptic systems involving critical nonlinearities and different Hardy-type terms ☆

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ABSTRACT

In this paper, a singular system of elliptic equations is investigated, which involves critical nonlinearities and deferent Hardy-type terms. Existence of positive solutions to the system is verified by variational methods and asymptotic properties of solutions at the singular point are established by the Moser iteration method. It is found that the functions in solutions have different singularities at the singular point.

1. Introduction

In this paper, we study the following elliptic system:

$$\begin{cases} -\Delta u - \mu_1 \dfrac{u}{|x|^2} = \left(|u|^q + |v|^q\right)^{\frac{2^*}{q}-1} |u|^{q-2}u + a_1 u + a_2 v & \text{in } \Omega, \\ -\Delta v - \mu_2 \dfrac{v}{|x|^2} = \left(|u|^q + |v|^q\right)^{\frac{2^*}{q}-1} |v|^{q-2}v + a_2 u + a_3 v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary such that $0 \in \Omega$, $a_i \in \mathbb{R}$, $i = 1, 2, 3$. Furthermore, the parameters satisfy the following assumption:

$(\mathcal{H}_1)$ $N \geq 4$, $2 \leq q \leq 2^* := \frac{2N}{N-2}$, $0 \leq \mu_2 \leq \mu_1 < \bar{\mu} := (\frac{N-2}{2})^2$.

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We work in the product space $H^2 := H \times H$, where $H := H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $(\int_\Omega |\nabla \cdot|^2 \, \mathrm{d}x)^{1/2}$. The corresponding energy functional of (1.1) is defined on H^2 by

$$J(u,v) := \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 - \mu_1 \frac{u^2}{|x|^2} - \mu_2 \frac{v^2}{|x|^2}\right)\mathrm{d}x - \frac{1}{2^*}\int_\Omega \left(|u|^q + |v|^q\right)^{\frac{2^*}{q}}\mathrm{d}x - \frac{1}{2}\int_\Omega \left(a_1 u^2 + 2a_2 uv + a_3 v^2\right)\mathrm{d}x. \tag{1.2}$$

Then $J \in C^1(H^2, \mathbb{R})$. The duality product between H^2 and its dual space $(H^2)^{-1}$ is denoted by $\langle J'(u,v), (\varphi,\phi)\rangle$, where $(u,v),(\varphi,\phi) \in H^2$ and $J'(u,v)$ is the Fréchet derivative of J at (u,v). A pair of functions $(u_0,v_0) \in H^2$ is said to be a solution of the problem (1.1) if

$$(u_0,v_0) \neq (0,0), \qquad \left\langle J'(u_0,v_0),(\varphi,\phi)\right\rangle = 0, \quad \forall(\varphi,\phi) \in H^2.$$

A solution of (1.1) is equivalent to a nonzero critical point of J. Standard elliptic argument shows that the solution (u_0,v_0) of (1.1) has the property:

$$u_0, v_0 \in C^2\left(\Omega \setminus \{0\}\right) \cap C^1\left(\bar{\Omega} \setminus \{0\}\right). \tag{1.3}$$

Problem (1.1) is related to the well-known Hardy inequality [12]:

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\,\mathrm{d}x \leq \frac{1}{\bar{\mu}}\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x, \quad \forall u \in C_0^\infty\left(\mathbb{R}^N\right),$$

which implies that the operator $L := (-\Delta \cdot - \mu\frac{\cdot}{|x|^2})$ is positive for all $\mu < \bar{\mu}$ and the following best constants are well defined for all $\mu, \mu_1, \mu_2 < \bar{\mu}$ and $0 < q \leq 2^*$:

$$S(\mu) := \inf_{u \in \mathcal{D}\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^2 - \mu\frac{u^2}{|x|^2})\mathrm{d}x}{(\int_{\mathbb{R}^N}|u|^{2^*}\mathrm{d}x)^{\frac{2}{2^*}}},$$
$$S(\mu_1,\mu_2) := \inf_{(u,v)\in\mathcal{D}\setminus\{(0,0)\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^2 + |\nabla v|^2 - \frac{\mu_1 u^2 + \mu_2 v^2}{|x|^2})\mathrm{d}x}{(\int_{\mathbb{R}^N}(|u|^q + |v|^q)^{\frac{2^*}{q}}\mathrm{d}x)^{\frac{2}{2^*}}}, \tag{1.4}$$

where $\mathcal{D} := D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to $(\int_{\mathbb{R}^N} |\nabla \cdot|^2 \, \mathrm{d}x)^{1/2}$.

In recent years, much attention has been paid to elliptic problems involving critical nonlinearities and the Hardy inequality and many results were obtained (e.g. [1,7–10,15,20]). In particular, it was verified that $S(\mu)$ has extremals of the type [20]:

$$V_\mu^\varepsilon(x) := \varepsilon^{\frac{2-N}{2}} U_\mu\left(\varepsilon^{-1}x\right), \quad \forall\varepsilon > 0, \quad \mu \in [0,\bar{\mu}), \tag{1.5}$$

where

$$U_\mu(x) = \left(\frac{2N(\bar{\mu}-\mu)}{\sqrt{\bar{\mu}}}\right)^{\frac{\sqrt{\bar{\mu}}}{2}} \left(|x|^{\frac{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}}{\sqrt{\bar{\mu}}}} + |x|^{\frac{\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}{\sqrt{\bar{\mu}}}}\right)^{-\sqrt{\bar{\mu}}}.$$

Elliptic systems involving critical nonlinearities and the Hardy inequality have been studied by some authors (e.g. [2,4,13,14,19]). It should be mentioned that, Hardy-type terms in the problems studied in [4,13,14,19] have the same coefficients and many ideas for non-singular elliptic systems can be applied directly.