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Compatibility conditions for Dirichlet and Neumann problems of Poisson's equation on a rectangle

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ABSTRACT

It is a well-known fact that the solution of Poisson's equation on a rectangle lacks regularity. Even for a smooth inhomogeneity, corner singularities arise in the derivatives of the solution. The very form of these singularities is of particular interest in numerical analysis; more precisely for the analysis of dimension splitting methods applied to parabolic equations. In this work, necessary and sufficient conditions on the inhomogeneity are derived which ensure a higher regularity of the solution of the Dirichlet or the Neumann problem – the so called compatibility conditions.

1. Introduction

For a bounded domain $\Omega \subset \mathbb{R}^2$ with $\mathcal{C}^\infty$ boundary $\partial\Omega$, the well-known *shift theorem* states that for any given $f \in H^k(\Omega)$ with $k \in \mathbb{N}_0$ the solution of the homogeneous Dirichlet problem of Poisson's equation

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1}$$

lies in $H^{k+2}(\Omega)$. This result can, for instance, be found in [4]. For $k \geq 1$ such a shift theorem does not hold, in general, for domains with corners such as a rectangle.

For the sake of simplicity, we will consider the unit square $\Omega = (0,1)^2$ in the following. Note that for $f \in L^2(\Omega)$, problem (1) has a unique solution in $H^2(\Omega) \cap H_0^1(\Omega)$, cf. [6, Theorem 3.2.1.2, p. 147]. To obtain a solution with higher regularity in the case of Poisson's equation on Ω, one has to impose conditions on the inhomogeneity f in the corners of the unit square Ω, which we will call *compatibility conditions*.

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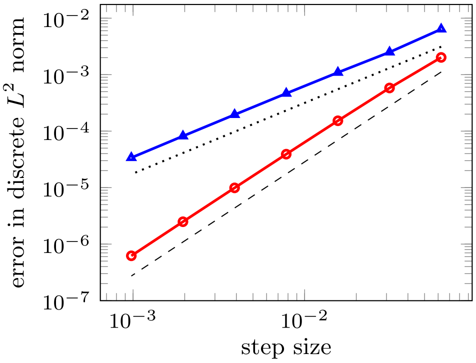

Fig. 1. Discrete L^2 error of the standard (blue triangles) and the modified exponential Strang splitting (red circles) applied to an evolution equation of the form (3) with an incompatible inhomogeneity g. The equation is discretized in space by standard finite differences with 100 grid points in each dimension. The dashed and dotted lines have slope 2 and 5/4, respectively.

For $f \in \bar{\mathcal{C}}^{2k}(\Omega)$ and $j \in \{1, \dots, k\}$, we set

$$C_j f = \sum_{i=1}^{j} (-1)^{i+1} \partial_x^{2j-2i} \partial_y^{2i-2} f.$$

Then the compatibility conditions read as

$$C_j f|_V = 0 \quad \text{for all } j = 1, \dots, k, \tag{2}$$

where V denotes the set consisting of the four vertices of the unit square Ω. In the classical setting, i.e. in Hölder spaces, these conditions were derived and investigated in [14]. Moreover, the regularity of the solution of elliptic boundary value problems on polygonal domains was investigated in Sobolev spaces in a rather general setting in [6] and in [2]. However, in the case of a rectangular domain, the above compatibility conditions can be derived and investigated in a very explicit manner. This is achieved by studying the regularity of corresponding quarter space problems. As such problems can be extended to the whole plane very easily, an explicit representation of the solution is obtained by the convolution of the extended inhomogeneity with a fundamental solution of the Laplacian. This representation of the solution is particularly suited to study its regularity which is the key to derive the compatibility conditions (2).

The explicit form of the compatibility conditions on a rectangle is of particular interest for constructing efficient dimension splitting methods which achieve full order of convergence when applied to inhomogeneous evolution equations in $L^2(\Omega)$ of the form

$$w'(t) = Lw(t) + g(t) \quad \text{for } t > 0, \qquad w(0) = w_0. \tag{3}$$

The involved operator $L = \partial_x(a\partial_x) + \partial_y(b\partial_y)$ is strongly elliptic with positive coefficients $a, b \in \bar{\mathcal{C}}^2(\Omega)$. Due to emerging corner singularities, standard dimension splitting methods suffer from an order reduction. Fig. 1 shows that a modification motivated by the investigation of the regularity of the stationary problem (1) leads to full order of convergence in case of the exponential Strang splitting. In Fig. 2, one observes that the pointwise error of the standard method is concentrated at the boundary whereas in case of the modified method it is strongly reduced and shifted towards the interior. For more details, we refer to [7] for the modification of dimension splitting methods and to [10,3] for the analysis of standard dimension splitting methods.

The main result of this work is the following proposition which we establish in Section 4.

Proposition 1. *For given $f \in H^{2k}(\Omega)$, the solution of (1) lies in $H^{2k+2}(\Omega)$ iff (2) holds.*