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Almost commuting orthogonal matrices

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ABSTRACT

We show that two almost commuting real orthogonal matrices are uniformly close to exactly commuting real orthogonal matrices. We prove the same for symplectic unitary matrices. This is in contrast to the general complex case, where not all pairs of almost commuting unitaries are close to commuting pairs. Our techniques also yield results about almost normal matrices over the reals and the quaternions. We conclude with an example where the K-theoretical obstructions to approximation cannot be avoided. Our example is inspired by the physical systems known as topological superconductors.

1. Introduction

In [14] Halmos asked if two almost commuting self-adjoint matrices are necessarily close to two exactly commuting self-adjoint matrices. To make this question as interesting as possible we take "almost" and "close" to be uniform across all matrix sizes. The question was answered in the positive by Lin [18], shortly thereafter Friis and Rørdam gave a short proof of Lin's Theorem in [12].

Before Lin's solution a lot of work went into investigating similar problems. Davidson showed in [5] that triples of almost commuting self-adjoint matrices need not be close to exactly commuting triples. Voiculescu showed that pairs of almost commuting unitary matrices are not necessarily close to pairs of exactly commuting unitaries [30]. Exel and the first named author gave a short proof of Voiculescu's result in [10]. The main idea in [10] is that if U, V are almost commuting unitaries then the winding number of the path in $\mathbb{C} \setminus \{0\}$ given by

$$t \mapsto \det\big((1-t)UV + tVU\big), \quad t \in [0,1], \tag{1}$$

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measures the obstruction to (U, V) being close to commuting unitaries. The winding number for a commuting pair is zero so the winding number also has to be zero for any pair that is close to a commuting pair.

The winding number is also referred to as the Bott index, a name that highlights its connection to K-theory. By [8,13] an almost commuting pair of unitary matrices U, V is close to a commuting pair of unitary matrices if and only if the Bott index of the pair is zero. For the C^*-algebraist the story of almost commuting unitary matrices could end here. However, if we concern ourselves with other scalars than the complex numbers, wish to measure noncommutativity by other means than the operator norm, and want algorithmic or quantitative results, there is much more to the story.

The recent focus in condensed matter physics on systems with time-reversal and other symmetries, see for instance [29], resulted in a new chapter of the story of almost commuting unitaries to be written. The unitary matrices U and V that arose in this context satisfied the new relations: $U^\tau = U$ and $V^\tau = V$ [23], where τ is either the transpose or the dual operation $\tau = \sharp$ given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^\sharp = \begin{pmatrix} D^{\mathrm{T}} & -B^{\mathrm{T}} \\ -C^{\mathrm{T}} & A^{\mathrm{T}} \end{pmatrix}. \tag{2}$$

Thus came an intense focus on symmetric unitary matrices and self-dual unitary matrices that almost commute. A new index, called the Pfaffian–Bott index, was introduced to measure the obstruction for pairs of self-dual unitaries to be close to commuting pairs. Building on the work in [23], we characterized when a pair of almost commuting self-dual or symmetric unitaries can be perturbed to an exactly commuting pair in [25]. From a physics point of view, this was a natural place to look.

The Pfaffian–Bott index is similar to the Bott index, and has turned out to be linked to the spin Chern number in certain two dimensional physical systems called topological insulators, see [15]. Loosely speaking, a topological insulator is a physical system that electricity can travel around but not through. Imagine, for instance, a shallow frozen lake that has melted around the boundary so ripples can form only along the boundary. Topological insulators generally have spin-momentum locking, so the flows of electrons are not ordinary currents. These specialized currents are hoped to lead to spintronic devices that will be more powerful than electronic devices, as they do not ignore the information in the spin.

Topological superconductors also exhibit a distinction between the behavior in the interior and on the boundary. Here the boundary states involve Majorana fermions [1] and are more difficult to describe. The potential applications of topological superconductors include topological quantum computation [27].

In the present paper we mostly continue the story of almost commuting unitary matrices in a different direction. We ask when a pair of almost commuting real-valued unitary matrices, i.e. real orthogonal matrices, are close to exactly commuting real-valued unitaries. Unlike in the complex case we find that a pair of almost commuting real orthogonals are always close to an exactly commuting pair. With very little extra work, we also get results for symplectic unitaries.

For a pair of real orthogonal matrices the winding number of the path Eq. (1) is always zero. By results from [8,13] this means that close to any pair of almost commuting real orthogonals there is a pair of exactly commuting unitary (though not necessarily real orthogonal) matrices. Unlike in the case of self-dual unitaries, it turns out that there is no new obstruction to finding real orthogonal approximants. Thus, our main theorem is:

Theorem 1.1. *For any $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever U and V are real orthogonal matrices in M_n with*

$$\|UV - VU\| \le \delta,$$