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# Piecewise polynomial approximation of a nonlocal phase transitions model

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ABSTRACT

Piecewise collocation-finite element and Galerkin-finite element methods are proposed and analysed for a nonlinear partial integro-differential equation that arises in the modeling of phase transitions. We compute solutions in both methods using some standard quadrature rules. We present the order of accuracy of such semidiscrete time dependent problem with full integral and quadrature for the Galerkin inner product considering both the real solutions and the approximate solutions are sufficiently smooth in whole domain $\Omega$. We also find an upper bound considering the approximate solutions are $L_2$ in $\Omega$ and $H^s$ in each subdomain $w_i$ such that $\Omega = \bigcup_i w_i$.



## 1. Introduction

A large number of real life problems in mathematical physics, neuro-sciences, mathematical biology, population dynamics, ecology and many other scientific areas can be modelled by a nonlinear partial integro-differential equation. Many models are non-local in space and continuous in both space and time and can be described as reaction-diffusion integral equations.

We consider the numerical approximation of such a nonlinear partial integro-differential equation [5,7,6,8,9,11,10,13,21,20,22,26,28,24,15,23,17]

$$u_t = \varepsilon\left(\int_\Omega J(x-y)u(y,t)dy - u(x,t)\int_\Omega J(x-y)dy\right) + f(u) \tag{1}$$

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Here the initial condition $u(x,0) = u_0(x)$, $x \in \Omega$ where $\Omega \subseteq \mathbb{R}^d$ $(d = 1,2,3)$, $f(u)$ is a bistable nonlinearity for the associated ordinary differential equation $u_t = f(u)$ which is taken to be

$$f(u) = (u-\alpha)\big(1-u^2\big), \quad -1 < \alpha < 1 \tag{2}$$

unless otherwise stated. $J(x-y)$ is a kernel function that models the interaction between particles at positions $x$ and $y$. The effect of close neighbours $x$ and $y$ is usually greater than that from more distant ones; this is incorporated in $J$. We assume that $J$ is a non-negative function satisfies the following conditions:

**H1** $J(x) \geq 0$;
**H2** $J(x)$ is normalised such that $\int_{\mathbb{R}^d} J(x)dx = 1$;
**H3** $J(x)$ is symmetric, i.e. $J(x) = J(-x)$, for all $x \in \mathbb{R}^d$;

Note that the convolution equation (1) is the $L_2$-gradient flow of the free energy functional [20,6]

$$E(u) = \varepsilon \int_\Omega \int_\Omega J(x-y)\big(u(y,t)-u(x,t)\big)^2 dxdy + \int_\Omega F\big(u(x,t)\big)dx \tag{3}$$

where $u$, and $J$ are defined above and $f(u) = -F'(u)$. In (3) the first integral penalises special variations in the solution and when $f(u)$ is given by (2) the second integral penalises states $u$ for which $|u| \neq 1$. If we consider a binary alloy, $u = +1$ in one pure phase and $u = -1$ in the other while values in between indicate a corresponding mixture of them. It is to be noted that the minima of $F$ have equal depth if $\int_{-1}^{1} f du = 0$. It is conceivable that a stable state $u$ exists taking values near $-1$ on an interval $(-\infty, a]$ and values near 1 on an interval of the form $[b,\infty)$ with a simple transition occurring on $[a,b]$. For more detail please see [22,17] and references therein.

Authors study some important properties of the model (1) in [19,21,20]. Duncan, Grinfeld and Stoleriu [20] inspect properties of stationary solutions, stability, coarsening of solutions and numerical approximation of (1) by piecewise polynomials. Duncan [21,20] shows results for the linear part and the full nonlinear IDE model in detail, and also compares the model with the famous Allen–Cahn model. Then he concentrates on approximation and stability and accuracy analysis of such piecewise approximations.

A detailed study on the continuity and the stability of steady state solutions of the model (1) can be found [11]. In this article authors find the relation between the continuity of solutions and $\varepsilon$. They conclude that the steady state solutions are continuous for large choices of $\varepsilon$ values. They also establish that the diffusion coefficient $\varepsilon$ determines the number and type of stable solutions.

Discontinuous Galerkin methods with interior penalties for elliptic and parabolic problems are of ongoing interest, and a substantial amount of work on local parabolic and elliptic type nonlinear problems were done by several authors [31,37,1,29] and references therein. Recently Wheeler, Riviere, Girault [31] and Shin, Ohm [29] introduced a locally conservative discontinuous Galerkin scheme for nonlinear local parabolic problems and derived various error estimates. Bounding the error for the nonlocal problem (1) is simpler than for standard parabolic problems because there are no space derivatives, and the discontinuous Galerkin average and jump terms are not needed.

The main purpose of this article is to present an essential amount of approximation using piecewise polynomials and mathematical theorems to support the accuracy of such approximations for the model (1) for spatial one dimensional case. This article deals with computational and theoretical analysis of the approximate solution of the nonlocal nonlinear parabolic problem (1). Various collocation and Galerkin type procedures using piecewise polynomials in space are considered for the approximation. In general, the piecewise polynomial trial functions used are discontinuous. We do not consider time discretisation here, instead we concentrate on semidiscrete schemes.