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Dirichlet problems on graphs with ends

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ABSTRACT

In classical potential theory, one can solve the Dirichlet problem on unbounded domains such as the upper half plane. These domains have two types of boundary points; the usual finite boundary points and another point at infinity. W. Woess has solved a discrete version of the Dirichlet problem on the ends of graphs analogous to having multiple points at infinity and no finite boundary. Whereas C. Kiselman has solved a similar version of the Dirichlet problem on graphs analogous to bounded domains. In this work, we combine the two ideas to solve a version of the Dirichlet problem on graphs with finitely many ends and boundary points of the Kiselman type.

1. Introduction

Let $D \subset \mathbb{R}^n$ be an open set and take a function $f: \partial D \to \mathbb{R}$. The Dirichlet problem on D with boundary data f is to find a unique function h which is harmonic on D, continuous on $\overline{D}$, and agrees with f on the boundary, i.e. $h|_{\partial D} = f$. Various methods, such as that of Perron, allow one to always find a harmonic function h which is associated with f in a natural way. The main challenge is to determine when (or where) this h matches f on ∂D. Given the way we've defined the Dirichlet problem, the continuity of f is clearly necessary. For bounded open sets D, the uniqueness criteria follows from the maximum principle for harmonic functions. However for unbounded domains uniqueness is non-trivial.

Consider the example of the upper half plane $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. If one wants to solve the Dirichlet problem on $\mathbb{H}$, we have trouble with uniqueness. Indeed, the functions $f_1(x, y) = y$ and $f_2(x, y) = 0$ are harmonic and identically equal to 0 on the boundary of the upper half plane. In classical potential theory, to obtain uniqueness one adds a point at infinity, $\hat{\mathbb{H}} = \mathbb{H} \cup \{\infty\}$, and then considers the Dirichlet problem on $\hat{\mathbb{H}}$ with boundary $\partial\hat{\mathbb{H}} = \partial\mathbb{H} \cup \{\infty\}$. Using Perron's method, to each $f: \partial\hat{\mathbb{H}} \to \mathbb{R}$, we can find a corresponding function $h: \hat{\mathbb{H}} \to \mathbb{R}$ which is harmonic. To ensure that this h equals f on $\partial\hat{\mathbb{H}}$ all that is required of f is to be continuous and bounded on $\partial\hat{\mathbb{H}}$, where continuous on $\partial\hat{\mathbb{H}}$ means that f is continuous on $\partial\mathbb{H}$ and continuous at the point infinity.

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In discrete potential theory, one often studies the Dirichlet problem on directed graphs. Here one considers an arbitrary at most countable set X as the vertex set. Weighted directed edges are provided by a structure function $\lambda : X \times X \to \mathbb{R}$, where $\lambda \geq 0$, and $\lambda(x, y) > 0$ denotes the existence of a directed edge from x to y with edge weight equal to $\lambda(x, y)$ ($\lambda(x, y) = 0$ implies that there is no edge from x to y), and with the additional condition $\sum_{y \in X} \lambda(x, y) = 1$, the graph has the structure of a Markov chain.

Considering the domain as a Markov chain allows one to attack these types of problems with either probability or analysis. This is of course true for the classical setting as well, e.g. [4]. Here we take the analytic approach.

In [7] Kiselman defines the boundary of X as

$$\partial X = \{x_0 \in X : \lambda(x_0, x_0) = 1\} \equiv \{x_0 \in X : \lambda(x_0, y) = 0 \text{ for all } y \in X \setminus \{x_0\}\}$$

and studies the Dirichlet problem in this setting. To obtain uniqueness results, he considers only finite graphs.

However in probability one often studies a *reversible* Markov chain, i.e. $\lambda(x, y) > 0 \iff \lambda(y, x) > 0$. In this setting Kiselman type boundary points trivially become isolated points of the graph. Thus the only interesting formulation of a Dirichlet problem is a problem at infinity. The principle works in this setting are those of [3,11,13], where they solve a Dirichlet problem where the boundary points are the ends of the graph.

The primary goal of this paper is to begin the study of discrete Dirichlet problems with mixed boundary type, i.e. a graph with both Kiselman type boundary points and ends. As a motivational example we look at the following simple 'discretized' version of the upper half plane.

Example 1. Consider the set $X = \mathbb{Z}^2 \cap \mathbb{H}$. Define

$$\lambda\big((x_1, y_1), (x_2, y_2)\big) = \begin{cases} 1/4 & : (x_2 - x_1)^2 + (y_2 - y_1)^2 = 1 \text{ and } y_1 \neq 0 \\ 1 & : x_1 = x_2,\ y_1 = y_2 \text{ and } y_1 = 0 \\ 0 & : \text{otherwise.} \end{cases}$$

Then $\partial X = \mathbb{Z}^2 \cap \{y = 0\}$ is a boundary set. It is easy to check that the functions $f_1(x, y) = y$ and $f_2(x, y) = 0$ are harmonic and identically equal to 0 on ∂X.

So uniqueness remains of principal concern. However in $\hat{X}$, the end compactification of X, we add a single point at infinity. Therefore $\hat{X}$ has the interesting structure of both a Kiselman type boundary and one end.

To study the Dirichlet problem on graphs of this type we introduce the notion of a quasi-reversible graph (reversible except for some Kiselman type boundary points). We prove (Theorem 2) a maximum principle with regard to a boundary of mixed type, from which follows (Theorem 3) a uniqueness result that holds for all connected quasi-reversible graphs.

It is a remarkable property of these mixed boundary problems that one-ended graphs are already non-trivial; a stark contrast to the setting in [3,13]. In the last section, we show (Theorem 5) there exists a unique solution to Dirichlet problems of mixed boundary types on one ended graphs. Finally we are able to extend the previous result to show (Theorem 6) there also exists a unique solution on graphs with finitely many ends. Analogously to the classical setting we require the boundary data to be continuous at infinity.

The Dirichlet problem, i.e. harmonic interpolation, on graphs has applications to coverage problems on topological sensor networks, [6, pp. 62–63] and shape description problems in digital image analysis [7,10].

2. Preliminaries

Introductions to various aspects of discrete potential theory can be found in [1,9,12]. The basic structure we will be working over can be thought of as a directed graph with weights on the edges, which is also