J. Math. Anal. Appl. 420 (2014) 1195–1209

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Interval maps quasi-symmetrically conjugate to a piecewise affine map

Huaibin Li [1]

School of Mathematics and Information Science, Henan University, Kaifeng 475004, People's Republic of China

ARTICLE INFO

Article history:
Received 4 December 2013
Available online 25 June 2014
Submitted by Richard M. Aron

Keywords:
Interval maps
Maximal entropy measure
Quasi-symmetry conjugacy
Semi-hyperbolicity

ABSTRACT

Consider a multimodal interval map f of C^3 with non-flat critical points. We establish several characterizations of the map f that is quasi-symmetrically conjugate to a piecewise affine map in the case f is topologically exact and all of its periodic points are hyperbolic repelling. In particular, we give a negative answer to a question posed by Henk Bruin in [1].

1. Introduction

Multimodal interval maps are often topologically conjugate to piecewise linear maps. So in a topological sense, such a multimodal map is the same as a piecewise linear map. In a metric sense, however, the difference is clear. Some metric similarities still occur, when the conjugacy satisfies certain constraints.

Let I be a compact interval of $\mathbb{R}$. A homeomorphism $h : I \to I$ on the interval is *quasi-symmetric* if there exists constant $K \geq 1$ such that for all $x \in I$ and all $\varepsilon > 0$ with $x \pm \varepsilon$ in I we have

$$\frac{1}{K} \leq \frac{|h(x+\varepsilon) - h(x)|}{|h(x) - h(x+\varepsilon)|} \leq K.$$

The notion of quasi-symmetry gained importance since Sullivan [19] proved the following rigidity result: Let $f_a(x) = 1 - ax^2$, if f_a and f_b are quasi-symmetrically conjugate and do not have a periodic attractor, then $a = b$.

E-mail address: lihbmath@henu.edu.cn.

[1] The author was partially supported by the National Natural Science Foundation of China (Grant No. 11101124) and the FONDECYT grant 3110060 of Chile.

http://dx.doi.org/10.1016/j.jmaa.2014.06.057

In this paper, we give several characterizations of a multimodal interval map that is quasi-symmetrically conjugate to a piecewise affine map. To state our main results, let us be more precise. Let I be a compact interval of $\mathbb{R}$. A non-injective continuous map $f : I \to I$ is *multimodal*, if there is a finite partition of I into intervals on each of which f is injective. A multimodal map $f : I \to I$ is *topologically exact*, if for every open subset U of I there is an integer $n \geq 1$ such that $f^n(U) = I$. A *turning point* of a multimodal map $f : I \to I$ is a point in I at which f is not locally injective. For a differentiable multimodal map $f : I \to I$, a point of I is *critical of* f if the derivative of f vanishes at it. In what follows, denote by $\mathrm{Crit}(f)$ the set of critical points of f, denote by $\mathrm{Crit}'(f)$ the turning points of f, and put $\mathrm{CV}(f) := f(\mathrm{Crit}(f))$. A C^1 multimodal map $f : I \to I$ is *of class C^3 with non-flat critical points*, if:

- The map f is of class C^3 outside $\mathrm{Crit}(f)$;
- For each critical point c of f there exist a number $\ell_c > 1$ and diffeomorphisms ϕ and ψ of $\mathbb{R}$ of class C^3, such that $\phi(c) = \psi(f(c)) = 0$, and such that on a neighborhood of c on I, we have $|\psi \circ f| = |\phi|^{\ell_c}$.

Recall that for an integer $n \geq 1$, a periodic point p of f of period n is *hyperbolic repelling* if $|Df^n(p)| > 1$, and that a critical point $c \in \mathrm{Crit}(f)$ is called *recurrent* if $c \in \omega(c)$, where $\omega(c)$ denote the ω-limit set of c that is the set of accumulation points of the forward orbit $\{f^n(c)\}_{n=0}^{+\infty}$ of c.

The *topological entropy* $h_{\mathrm{top}}(f)$ *of* f is equal to the supremum of the metric entropies $h_\mu(f)$ taken over all f-invariant Borel probability measures μ, see for example [20]. An f-invariant Borel probability measure μ such that $h_\mu(f) = h_{\mathrm{top}}(f)$ is called a *maximal entropy measure.* It is well-known that a multimodal interval map $f : I \to I$ that is topologically exact has a unique maximal entropy measure μ_f, see for example [4]. Moreover, μ_f is non-atomic.

For a point x in I, $r > 0$, an integer $m \geq 1$, and each j in $\{0, 1, \cdots, m-1\}$, let W_j be the pull-back of $B(f^m(x), r) \cap I$ by f^{m-j} containing $f^j(x)$. The *criticality of f^m at x with respect to r* is defined as the following number

$$\#\big\{j \in \{0, 1, \cdots, m-1\} : W_j \cap \mathrm{Crit}'(f) \neq \emptyset\big\}.$$

Moreover, the map f is said to be *semi-hyperbolic*, if there exist constants $r > 0$ and $D \geq 1$ such that for every x in I and each integer $n \geq 1$ the criticality of f^n at x with respect to r is at most D.

Definition 1. A Borel measure μ on a metric space (X, dist) is said to be *doubling*, if there are constants $C_* > 0$ and $r_* > 0$ such that for each x in X and r in $(0, r_*)$ we have

$$\mu\big(B(x, 2r)\big) \leq C_* \mu\big(B(x, r)\big).$$

See for example [3] for background on doubling measures.

The main result of this paper is the following theorem.

Theorem 1. *Let $f : I \to I$ be a multimodal map of class C^3 with non-flat critical points and with all periodic points hyperbolic repelling. If f is topologically exact, then the following statements are equivalent.*

(1). f is semi-hyperbolic;
(2). f has no recurrent critical points;
(3). The maximal entropy measure of f is doubling;
(4). f is quasi-symmetrically conjugate to a piecewise affine function with slope equal to $\pm\exp(h_{\mathrm{top}}(f))$.