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Explicit solutions for an optimal stock selling problem under a Markov chain model

Q. Zhang

Department of Mathematics, University of Georgia, Athens, GA 30602, United States

A R T I C L E   I N F O

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A B S T R A C T

This paper is concerned with explicit solutions for a classical optimal stock selling problem. In contrast to almost all market models treated in the literature, the underlying market is solely determined by a two-state Markov chain. Such Markov chain model is strikingly simple and yet appears capable capturing various market movements ranging from close-to-Brownian motion to no-so-Brownian ones. The purpose of this paper is to study the optimal selling rule under such a model and develop a set of analysis techniques useful for related optimal stopping problems. In this paper, the goal of the problem under consideration is to find an optimal stopping time to sell the stock so as to maximize an expected return. Explicit solutions to the associated variational inequalities are obtained. These solutions are given in terms of a set of threshold levels. Verification theorems are provided to justify their optimality. Finally, numerical examples are provided to illustrate the results.

1. Introduction

Most market models in the literature are Brownian motion based including geometric Brownian motion, diffusion with possible jumps and regime switching; see related books by Duffie [3], Hull [9], Elliott and Kopp [4], Fouque et al. [5], Karatzas and Shreve [10], and Musiela and Rutkowski [13] among others. An alternative is the binomial tree model introduced by Cox–Ross–Rubinstein. The BTM is widely used in option pricing thanks to its striking simplicity and clear advantage when pricing American type options. However, a main drawback of the BTM is its non-Markovian nature. The lack of Markovian property makes it difficult to work with mathematically, not to mention closed-form solutions. To preserve the BTM's simplicity and enhance its mathematical tractability, we propose a model solely determined by a two-state Markov chain. The Markov chain model appears to be natural for not frequently traded securities such as illiquid stocks. Its capability capturing this type of markets makes it preferable in related applications. In

E-mail address: qingz@math.uga.edu.

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addition, it is closely related to the traditional geometric Brownian motion models when the jump rates are large.

Recently, several Markov chain based models are developed. For example, van der Hoek and Elliott [17] introduced a stock price model based on stock dividend rates and a Markov chain noise. Norberg [14] used a Markov chain to represent interest rate and considered a market model driven by a Markov chain. In particular, the market model in [14] resembles a GBM in which the 'drift' is approximated by the duration between jumps and the 'diffusion' is given in terms of jump times. An additional advantage of a Markov chain driven model is that its price is almost everywhere differentiable. Such differentiability is desirable in the optimal control type market analysis proposed by Barmish and Primbs [1]. In connection with dynamic programming problems, the corresponding Hamilton–Jacobi–Bellman (HJB) equations are of the first order, which are easier to analyze than those under traditional Brownian motion based models. Finally, the Markov chain model is not that far apart from a GBM because it can be used to approximate a GBM by varying its jump rates. In fact, it is shown in Example 1 that a properly scaled Markov chain model converges weakly to that of a GBM as the jump rates go to infinity.

When to sell a stock is a crucial component in stock trading. It determines when to take profits or to cut losses. It is probably the most emotional part for individual investors in the trading process. Selling rules in financial markets have been studied for many years. For example, Zhang [21] considered a selling rule determined by two threshold levels: a target price and a stop-loss limit. One makes a selling decision whenever the price reaches either levels. Under a switching GBM, the objective is to determine these threshold levels to maximize an expected discounted reward function. In [21], such optimal threshold levels are obtained by solving a set of two-point boundary value problems. In Guo and Zhang [6], they considered the optimal selling rule under a GBM model with regime switching. Using a smooth-fit technique, they were able to convert the optimal stopping problem to a set of algebraic equations. These algebraic equations were used to determine the optimal target levels. In addition to these analytical results, various mathematical tools have been developed to compute these threshold levels. For example, a stochastic approximation technique was used in Yin, Liu and Zhang [18] and a linear programming approach was developed in Helmes [7]. In addition, Merhi and Zervos [12] studied an investment capacity expansion/reduction problem following a dynamic programming approach under a GBM market model. Similar problem under a more general market model was treated by Løkka and Zervos [11]. In addition, Du Toit and Peskir [2] considered selling at the peak problem under a geometric Brownian motion and Henderson and Hobson [8] treated a risk-averse selling under a stopping/optimal control framework. General treatment in connection with optimal stopping and finance applications can be found the book by Peskir and Shiryaev [16].

The purpose of this paper is study a classical optimal stopping problem under a simple yet flexible Markovian model and to develop a set of mathematical analysis techniques for Markov chain problems. In this paper, the stock price is assumed to follow a Markov chain model. Under this model, the state of the Markov chain can be estimated based on the stock price increments. This makes the Markov chain observable. In addition to its simplicity, the Markov chain model is able to capture price movements of a broader range of stocks. In this paper, under the Markov chain model, we consider an optimal stock selling rule and obtain its solution in terms of a set of threshold levels. In particular, we solve the corresponding dynamic programming problem and obtain these threshold levels. We point out that the standard smooth-fit method that works in a GBM setting is not adequate in one of the cases in this paper because of the lack of enough equations for the unknown parameters. To solve the problem, we need to explore other convexity conditions to determine uniquely these parameters. We also provide a set of sufficient conditions that guarantee their optimality. Numerical examples are reported to illustrate these results.

This paper is organized as follows. In Section 2, we formulate the problem and make a few assumptions. In Section 3, we study properties of the value functions, the associate HJB equations, and their solutions. In Section 4, we provide a set of sufficient conditions that guarantee the optimality of our selling rule. We