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A class of non-integrable systems admitting an inverse integrating factor

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ABSTRACT

We study the existence of an inverse integrating factor for a class of systems, in general non-integrable, whose lowest-degree quasi-homogeneous term is a Hamiltonian system and its Hamiltonian function only has simple factors over $\mathbb{C}[x,y]$.

1. Introduction and statement of the main results

We consider an autonomous system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \big(P(\mathbf{x}), Q(\mathbf{x})\big)^T, \tag{1}$$

whose origin is an equilibrium point and $P, Q \in \mathbb{C}[\![x,y]\!]$ (algebra of the power series in x and y with coefficient in $\mathbb{C}$) defined in a neighborhood of the origin $U \subset \mathbb{C}^2$.

A function f (or a curve $f(x,y) = 0$) with $f \in \mathbb{C}[\![x,y]\!]$ non-null, is an invariant function (or an invariant curve) of system (1) on U, if there is $K \in \mathbb{C}[\![x,y]\!]$ such that $L_{\mathbf{F}} f = Kf$, being $L_{\mathbf{F}} f := \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q$. A function K is named the cofactor of the invariant curve $f = 0$.

A non-null function $V \in \mathbb{C}[\![x,y]\!]$ is an inverse integrating factor of system (1) on U if $V = 0$ is an invariant curve of system (1) whose cofactor is the divergence of the vector field, i.e. $L_{\mathbf{F}} V = \operatorname{div}(\mathbf{F})V$, being $\operatorname{div}(\mathbf{F}) := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$.

In this work, our aim is to provide conditions on the system in order to study the existence of an inverse integrating factor.

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It is known that if system (1) has a formal inverse integrating factor, which is non-zero at origin, then the system (1) is formally integrable at origin. Therefore, if system (1) is not formally integrable at origin and it has an inverse integrating factor V, then $V(\mathbf{0}) = 0$. For more details about the relation between the integrability and the inverse integrating factor see [3,4].

The presence of an inverse integrating factor is also related to the existence of a limit cycle and its hyperbolicity and cyclicity, see [5–12].

Given $\mathbf{t} = (t_1, t_2)$ non-null with t_1 and t_2 non-negative integer numbers without common factors, we denote by $\mathcal{P}_k^{\mathbf{t}}$ the vector space of quasi-homogeneous polynomials of type $\mathbf{t}$ and degree k, i.e.

$$\mathcal{P}_k^{\mathbf{t}} = \{f \in \mathbb{C}[x,y] : f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^k f(x,y)\},$$

and by

$$\mathcal{Q}_k^{\mathbf{t}} = \{\mathbf{F} = (P,Q)^T : P \in \mathcal{P}_{k+t_1}^{\mathbf{t}},\ Q \in \mathcal{P}_{k+t_2}^{\mathbf{t}}\}$$

the vector space of the quasi-homogeneous polynomial vector fields of type $\mathbf{t}$ and degree k. Any vector field is expanded into quasi-homogeneous terms of type $\mathbf{t}$ of successive degrees. Thus, the vector field $\mathbf{F}$ becomes

$$\mathbf{F} = \mathbf{F}_r + \mathbf{F}_{r+1} + \cdots,$$

for some $r \in \mathbb{Z}$, where $\mathbf{F}_j = (P_{j+t_1}, Q_{j+t_2})^T \in \mathcal{Q}_j^{\mathbf{t}}$ and $\mathbf{F}_r \not\equiv \mathbf{0}$. Such expansions are expressed by $\mathbf{F} = \mathbf{F}_r + \text{q-h.h.o.t.}$

If we select the type $\mathbf{t} = (1,1)$, we are using in fact the Taylor expansion, but in general, each term in the above expansion involves monomials with different degrees. We cite some properties, see [2,3].

- $\mathbf{D}_0 := (t_1x, t_2y)^T \in \mathcal{Q}_0^{\mathbf{t}}$,
- if $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$, $|\mathbf{t}| = t_1 + t_2$, then $\mathbf{X}_h := (-\partial h/\partial y, \partial h/\partial x)^T \in \mathcal{Q}_r^{\mathbf{t}}$,
- every $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$ can split as

$$\mathbf{F}_k = \mathbf{X}_h + \mu\mathbf{D}_0 \tag{2}$$

  with $h = (\mathbf{D}_0 \wedge \mathbf{F}_k)/(k+|\mathbf{t}|)$ and $\mu = \operatorname{div}(\mathbf{F}_k)/(k+|\mathbf{t}|)$. This sum is known as *the conservative–dissipative splitting of a quasi-homogeneous vector field*.

Given a type $\mathbf{t}$ and $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$, we consider the systems which are formally orbital equivalent to $\dot{\mathbf{x}} = \mathbf{X}_h + \mu\mathbf{D}_0$ with $\mu = \mu_r + \text{q-h.h.o.t.}$ and $\mu_r \in \mathcal{P}_r^{\mathbf{t}}$. The following result characterizes them.

Theorem 1. *Given $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$, a system $\dot{\mathbf{x}} = \mathbf{X}_h +$ q-h.h.o.t. is formally orbital equivalent to $\dot{\mathbf{x}} = \mathbf{X}_h + \mu\mathbf{D}_0$ with $\mu = \sum_{j>r}\mu_j$, $\mu_j \in \mathcal{P}_j^{\mathbf{t}}$, if and only if it has an invariant curve $f = 0$ of the form $f = h +$ q-h.h.o.t. with f a function conjugate to h (i.e. there exists a formal diffeomorphism Φ such that $h = f \circ \Phi$).*

In this paper, we deal with a class of systems of the form

$$\dot{\mathbf{x}} = \mathbf{X}_h + \text{q-h.h.o.t.}, \tag{3}$$

where h is a quasi-homogeneous polynomial, whose factorization on $\mathbb{C}[x,y]$ only has simple factors. We note that this condition on h is generic.

We cite the results obtained in the paper. We provide a formal orbital equivalent normal form of system (3), i.e. an expression of this system after a change of state variables and a re-parameterization of the