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Large solutions to semi-linear elliptic systems with variable exponents

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ABSTRACT

We consider the elliptic system $\Delta u = u^a v^b$, $\Delta v = u^c v^d$ where the exponents are non-negative spherically symmetric functions. We study positive solutions on balls of finite and infinite radius R which satisfy $\lim_{|x|\to R} u(x) = \lim_{|x|\to R} v(x) = \infty$. We give conditions on the exponents that either ensure existence or nonexistence of these large solutions.

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1. Introduction

We consider the elliptic system

$$\begin{cases} \Delta u = u^{a(|x|)} v^{b(|x|)} & \text{in } B_R \\ \Delta v = u^{c(|x|)} v^{d(|x|)} & \text{in } B_R \end{cases} \tag{1.1}$$

where B_R is a ball of radius R in $\mathbb{R}^N$ (bounded or unbounded) centered at the origin and the exponents a, b, c and d are nonnegative continuous radial functions. We establish existence and nonexistence results for positive large solutions; i.e., positive solutions (u, v) that satisfy

$$u(x) \to \infty \quad \text{and} \quad v(x) \to \infty \quad \text{as } |x| \to R. \tag{1.2}$$

(If $R = \infty$, then $B_R = \mathbb{R}^N$, and the limit in (1.2) should be taken as $|x| \to \infty$.) Solutions of (1.1) that satisfy (1.2) are called large solutions of (1.1) on B_R. In particular, if $R = \infty$, solutions are called entire large solutions.

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Large solutions for elliptic systems have a relatively short history starting with [7] by the first author and Wood followed by extensions in various directions (see, e.g., [1,2,5,8,10–12]). None of these nor any other articles, however, consider variable exponents as we do here.

In this article we consider the same system as García-Melián and Rossi's [2] with important differences, not the least of which is that their exponents are constant unlike ours. On the other hand, they consider bounded smooth domains where we consider only balls (except for Theorem 3.3). Using the notation

$$\mathcal{D}(r) := \det\big(A(r) - I\big), \qquad A(r) := \begin{bmatrix} a(r) & b(r) \\ c(r) & d(r) \end{bmatrix}, \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.3}$$

they establish large solutions for the subcritical case ($\mathcal{D} > 0$) and critical case ($\mathcal{D} = 0$) with $\mathcal{D}$ constant. Here we consider, in addition to these cases, the supercritical case $\mathcal{D}(r) \le 0$. They also restrict a and d so that $\min\{a, d\} > 1$, but we consider that case as well as $\max\{a, d\} \le 1$. We show that the system admits a large solution if $\mathcal{D}(r) < 0$ and $0 \le a, d \le 1$ (Theorem 3.1). On the other hand, we show that if $\mathcal{D}(r) > 0$ and $0 \le a, d < 1$ on the boundary of the ball, then the system has no large solution on the ball (Theorem 3.2). For entire large solutions we show such a solution exists if $\mathcal{D}(r) \ge 0$ with $bc > 0$ and $0 \le a, d < 1$ (Theorem 4.3). Conversely, we show that if either $\max\{a, d\} > 1$ at infinity, or $a < b+1$, $d < c+1$, and $\limsup_{r\to\infty} \mathcal{D}(r) < 0$, then (1.1) has no positive entire radial solution (Theorems 4.4 and 4.5). Finally, we note that existence of large solutions for a single equation with variable exponents was considered in [3,6], and [9].

The paper is organized as follows. In Section 2, we state and prove some basic results that will be used later. In Section 3, we consider existence and nonexistence of large positive solutions on balls of finite radius. In Section 4, existence and nonexistence of entire large solutions are treated. We conclude that section and the article by showing (Corollary 4.6) that a combination of our results gives a natural extension to García-Melián and Rossi's [2] work on bounded domains to provide a necessary and sufficient condition for (1.1) to have an entire large solution. For convenience we have assumed throughout the paper that $N > 2$. However, all results with proofs that use this restriction remain valid for $N = 1, 2$ with minor changes in the proofs.

2. Preliminaries

This section will be devoted to establishing some preliminary results. These will be used in proving existence as well nonexistence of large radial solutions on balls of finite radius, or in $\mathbb{R}^N$.

Lemma 2.1. *Suppose $R > 0$, and $b(r)c(r) > (1-a(r))(1-d(r))$ on the closed interval $[0, R]$. Given $r \in [0, R]$ there is $\varepsilon > 0$ such that $\beta_r\gamma_r > (1-\alpha_r)(1-\delta_r)$, where*

$$\alpha_r := \min_{N(r)} a(s), \qquad \beta_r := \min_{N(r)} b(s), \qquad \gamma_r := \min_{N(r)} c(s), \quad \text{and} \quad \delta_r := \min_{N(r)} d(s), \tag{2.1}$$

and $N(r) := [r-\varepsilon, r+\varepsilon] \cap [0, R]$. Similarly, if $b(r)c(r) < (1-a(r))(1-d(r))$ then given $r \in [0, R]$ there is an $\varepsilon > 0$ such that $\beta_r\gamma_r < (1-\alpha_r)(1-\delta_r)$, where

$$\alpha_r := \max_{N(r)} a(s), \qquad \beta_r := \max_{N(r)} b(s), \qquad \gamma_r := \max_{N(r)} c(s), \quad \text{and} \quad \delta_r := \max_{N(r)} d(s), \tag{2.2}$$

and $N(r) := [r-\varepsilon, r+\varepsilon] \cap [0, R]$.

Proof. We prove only the case $bc > (1-a)(1-d)$ since the case $bc < (1-a)(1-d)$ is virtually the same. Let us fix $r \in [0, R]$, and for a given $\rho > 0$, let $B(\rho) := [r-\rho, r+\rho] \cap [0, R]$. We introduce the constants