J. Math. Anal. Appl. 420 (2014) 1500–1532

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Existence and uniqueness of weak solutions to Ginzburg–Landau equation with external noise and stochastic perturbation ☆

Jinlong Wei, Bin Liu *

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, Hubei, PR China

ARTICLE INFO

Article history:
Received 20 January 2013
Available online 26 April 2014
Submitted by U. Stadtmueller

Keywords:
Stochastic Ginzburg–Landau equation
Transport equation
Weak solution
Existence
Uniqueness

ABSTRACT

In this paper, the Ginzburg–Landau and fractional Ginzburg–Landau equations with external noise and stochastic perturbations are studied, and we obtain the global existence and uniqueness of weak L^1-solutions. Our proof is supported by the generalized Itô formula and a careful analysis of the nonlinear transport equation. After founding the well-posedness for the nonlinear transport problem in the class of L^p $(1 < p \leq \infty)$, we also gain the well-posedness of the Ginzburg–Landau and fractional Ginzburg–Landau equations in the same spaces, correspondingly.

1. Introduction

As a basic model of nonlinear phenomena, the Ginzburg–Landau (GL) equation is used in various areas of physics and chemistry to describe, for example, open flow motions, equilibria, traveling waves in binary fluid mixtures and spatially extended nonequilibrium systems. Besides, in [9], it is also used as a model equation, simpler than the Navier–Stokes equation, to study turbulence phenomena.

However, in the derivation of these ideal models, some perturbations may be neglected, such as molecular collisions in gases and liquids. For this reason, it is necessary to add a reasonable amount of noise to this equation. Considering the noise, it is interesting to study the stochastic Ginzburg–Landau (SGL) equation with a potential term as given below:

$$\frac{\partial}{\partial t}u(t,x) - \left(\frac{1}{2} + i\lambda\right)\Delta u(t,x) + (\gamma + i\eta)|u|^{2\sigma}u(t,x) + cu(t,x) + g(t,x,u,\nabla u)\dot{W}_t = f(t,x), \tag{1.1}$$

☆ This work was partially supported by NNSF of China (Grant No. 11171122).
* Corresponding author.
E-mail address: binliu@mail.hust.edu.cn (B. Liu).

http://dx.doi.org/10.1016/j.jmaa.2014.04.051
0022-247X/

in $(0,T)\times O$, for simplicity, where λ, γ, η, σ and c are given real numbers, $c>0$ is the instability parameter and λ the dispersive parameter, and O is an open domain of $\mathbb{R}^d$ with sufficiently regular boundary.

Note that a Stratonovich product is meaningful in (1.1) only for spatially corrected noise. Here, we wish to study uncorrelated noise so that the product in (1.1) is understood in the sense of Itô. To formulate the initial-boundary value problem, we assign initial and boundary conditions

$$u(0,x)=u_0(x), \tag{1.2}$$

$$u|_{\partial O}=h(t,x), \tag{1.3}$$

where u_0 is a function from O to $\mathbb{R}$ and h is a function from $(0,T)\times\partial O$ to $\mathbb{R}$.

A rigorous way to comprehend (1.1), (1.2), (1.3) is to establish the existence and uniqueness result for solutions in proper spaces. It should be noted that, as a parabolic type stochastic partial differential equation, most existing studies focus on O bounded, which is essential in general to use the Galerkin method or the fixed point method. For example, in [13], Smith considered (1.1), (1.2), (1.3) for $g(t,x,u,\nabla u)=g(t,x)$, W a cylinder Wiener process and O an open bounded domain; by the standard energy method, he derived the existence of solutions in an L^2 setting.

When g is u-dependent, depending solely on u, in [14], Smith discussed (1.1), (1.2) with $O=\mathbb{R}$ and $g=g(u)$ belonging to Lipschitz function space; under the periodic assumption on (1.2), he proved the well-posedness of solutions in $E_T=\mathcal{C}([0,T];L^{2k}(I))\cap L^{p_0}((0,T)\times I)$, where $p_0=2\sigma+2k$, $k\geq 1$, $I=[0,1]$. Similarly, in [12], Odasso studied (1.1), (1.2), (1.3) with $h=0$ and obtained the existence of weak solutions in $L^2([0,1]^d)$ under certain hypotheses on g, including L^∞-bounds and globally Lipschitz. For more details on this topic, one can consult [13,14,12] and related references cited therein.

It is remarked that, without the stochastic perturbations, (1.1) has been studied by many researchers, see e.g. [5,11] and the references therein. Most of the papers mentioned here concentrate their attention on the dynamic behavior of the GL equation in the $L^2(O)$ setting (O bounded) without external noise, such as the global existence of solutions [5] and the stability of radial solutions [11].

Using a different philosophy, in [3], Deissler discussed the GL equation

$$\frac{\partial}{\partial t}u-\left(\frac{1}{2}+i\lambda\right)\frac{\partial^2 u}{\partial X^2}+(\gamma+i\eta)|u|^2u+cu=0, \tag{1.4}$$

in the presence of low-level external noise in $\mathbb{R}$, where $X=x-vt$ ($x\in\mathbb{R}$) is a coordinate in the frame of reference moving at the group velocity v. Transforming (1.4) to the stationary frame gives

$$\frac{\partial}{\partial t}\bar{u}-\left(\frac{1}{2}+i\lambda\right)\frac{\partial^2 \bar{u}}{\partial x^2}+(\gamma+i\eta)|\bar{u}|^2\bar{u}+v\frac{\partial\bar{u}}{\partial x}+c\bar{u}=0, \tag{1.5}$$

where $\bar{u}$ is the transformed variable. Then he studied the numerical solutions of (1.5) and compared the three types of dynamic behavior of (1.4) and (1.5) when (1.5) has a nonzero group velocity (such as plane Poiseuille flow [15]) on the background of $L^1(\mathbb{R})$.

Being directly inspired by [3], as a primary purpose of the present paper, we plan to study the dynamic behavior of the GL equation in a general unbounded domain in $\mathbb{R}^d$ (including $\mathbb{R}^d$) with external noise and stochastic perturbations. First, we establish the fundamental existence and uniqueness result on solutions, and then we discuss the asymptotic behaviors of solutions as t tends to infinity. For more about this topic, the reader is referred to [12,16,17] and the references cited therein.

In this paper, letting $O=\mathbb{R}^d$, we wish to found the existence and uniqueness of solutions globally in the setting of L^1 for any $u_0\in L^1$. To achieve this goal, we confine our discussion to

$$g(t,x,u,\nabla u)=\nabla u,\qquad \lambda=\eta=0 \tag{1.6}$$