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An infinite-dimensional linking theorem without upper semi-continuous assumption and its applications

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ABSTRACT

We obtain a new infinite-dimensional linking theorem. This theorem replaces the upper semi-continuous assumption in the classical infinite-dimensional linking theorem of Kryszewski and Szulkin by a new assumption. As an application of this theorem, we obtain a nontrivial solution for a strongly indefinite periodic Schrödinger equation with sign-changing nonlinearity.

1. Introduction and statement of results

The infinite-dimensional linking theorems in critical point theory are powerful tools for studying the existence of critical points of strongly indefinite functionals. These functionals arise for example in the study of standing wave solutions of periodic Schrödinger equations, homoclinic orbits of Hamiltonian systems, entire solutions of elliptic systems in whole spaces and standing wave solutions of Dirac equations, see for instance [1–8,10,11,14–24]. The first such theorem was established by Kryszewski and Szulkin in [7] (see also [20, Chapter 6]) and was used to obtain nontrivial solutions for superlinear periodic Schrödinger equations. And in [10] and [11], the authors used this theorem to obtained nontrivial solutions for periodic Schrödinger equations with more general nonlinearities. This theorem was also used by Szulkin and Zou [16] to obtain homoclinic orbits for asymptotically linear Hamiltonian system. Several infinite-dimensional linking theorems different from but related to Kryszewski and Szulkin's were also found in the past ten years. We mention the book of Zou and Schechter [24] and the book of Ding [5], where these theorems were established and were applied to study various strongly indefinite problems.

To state the theorem of Kryszewski and Szulkin, we need some notations.

Let X be a separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. $X^{\pm}$ are closed subspaces of X and $X = X^+ \oplus X^-$. Let $\{e_k^-\}$ be the total orthonormal sequence in X^-. Let

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$$Q : X \to X^+, \qquad P : X \to X^- \tag{1.1}$$

be the orthogonal projections. We define

$$|||u||| = \max\left\{\|Qu\|, \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \left|\left(Pu, e_k^-\right)\right|\right\} \tag{1.2}$$

on X. Then

$$\|Qu\| \le |||u||| \le \|u\|, \quad \forall u \in X.$$

And if $\|u_n\|$ is bounded, then $|||u_n - u||| \to 0 \Leftrightarrow u_n \rightharpoonup u$ in X (see Remark 6.1 of [20]). The topology generated by $|||\cdot|||$ is denoted by τ, and all topological notations related to it will include this symbol.

Let $R > r > 0$ and $u_0 \in X^+$ with $\|u_0\| = 1$. Set

$$N = \left\{u \in X^+ \,\middle|\, \|u\| = r\right\}, \qquad M = \left\{u + tu_0 \,\middle|\, u \in X^-,\ t \ge 0,\ \|u + tu_0\| \le R\right\}. \tag{1.3}$$

Then, M is a submanifold of $X^- \oplus \mathbb{R}^+ u_0$ with boundary

$$\partial M = \left\{u \in X^- \,\middle|\, \|u\| \le R\right\} \cup \left\{u + tu_0 \,\middle|\, u \in X^-,\ t > 0,\ \|u + tu_0\| = R\right\}. \tag{1.4}$$

In the celebrate paper [7], Kryszewski and Szulkin proved the following theorem (see also [20, Theorem 6.10]):

Theorem 1.1. *(See [7].) If $H \in C^1(X, \mathbb{R})$ satisfies:*

(a) *H is τ-upper semi-continuous, i.e., for any $a \in \mathbb{R}$, $H_a := \{u \in X \mid H(u) \ge a\}$ is a τ-closed set.*
(b) *H' is weakly sequentially continuous, i.e., if $u \in X$ and $\{u_n\} \subset X$ are such that $u_n \rightharpoonup u$, then, for any $\varphi \in X$, $\langle H'(u_n), \varphi\rangle \to \langle H'(u), \varphi\rangle$. And*

$$\sup_M H < +\infty.$$

(c) *There exist $u_0 \in X^+$ with $\|u_0\| = 1$, and $R > r > 0$ such that*

$$\inf_N H > \sup_{\partial M} H.$$

Then there exist $c \in [\inf_N H, \sup_M H]$ and a $(PS)_c$ sequence $\{u_n\} \subset X$ for H, i.e., $\{u_n\}$ satisfies

$$H(u_n) \to c, \qquad H'(u_n) \to 0. \tag{1.5}$$

Under the assumptions of this theorem, the dimension of X^- can be infinite. Therefore, it is a generalization of the classical finite-dimensional linking theorem (see, for example, [20, Theorem 2.12]). And it has been extensively applied to study various variational problems with strongly indefinite structures, i.e., functionals of the form

$$H(u) = \frac{1}{2}\langle Lu, u\rangle - \psi(u) \tag{1.6}$$

defined on a Hilbert space X, where $L : X \to X$ is a self-adjoint operator with negative and positive eigenspaces both infinite-dimensional. The study of such functionals is motivated by a number of problems from mathematical physics, see for instance [1–8,10,11,14–24].