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The connectedness of some two-dimensional self-affine sets

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ABSTRACT

In the paper, we mainly discuss the connectedness of two kinds of self-affine sets. One is generated by matrix $A = \begin{pmatrix} p & 0 \\ -a & q \end{pmatrix}$ and digit set $\mathcal{D} = \{(is, jt)^T : i = 0, 1, \ldots, |q| - 1, j = 0, 1, \ldots, |p| - 1\}$, where $s, t \neq 0$ and $p, q \in \mathbb{Z}$ with $3 \leq |p| + 1 < |q| < 2|p| - 1$. The other is generated by matrix $A = \begin{pmatrix} p & 0 \\ -a & q \end{pmatrix}$ and digit set $\mathcal{D} = \{(is, (di + j)t)^T : i = 0, 1, \ldots, |p| - 1, j = 0, 1, \ldots, |q| - 1\}$, where $s, t \neq 0$, and $p, q, d \in \mathbb{Z}$ with $|p|, |q| \geq 2$. The sufficient or necessary conditions for their connectedness are revealed.

1. Introduction

Let A be an $n \times n$ expanding matrix (all its eigenvalues have moduli > 1). Let $\mathcal{D} = \{d_1, d_2, \ldots, d_m\} \subset \mathbb{R}^n$ be a finite set of m distinct vectors, called an *m-digit set*. Then linear maps

$$S_i(x) = A^{-1}(x + d_i), \quad 1 \leq i \leq m, \tag{1.1}$$

are all contractive with respect to a suitable metric on $\mathbb{R}^n$, and it is well known [10] that there exists an unique non-empty compact set T satisfying the set-valued functional equation

$$T = \bigcup_{i=1}^{m} S_i(T). \tag{1.2}$$

Then T can be explicitly given by

$$T := T(A, \mathcal{D}) := \left\{ \sum_{i=1}^{\infty} A^{-i} e_i : e_i \in \mathcal{D} \right\}. \tag{1.3}$$

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T is called a *self-affine set* generated by the *iterated function system* (IFS) $\{S_i\}_{i=1}^m$. Moreover, it is called a *self-affine tile* if it has nonempty interior and $m = |\det A|$, where $\det A$ is the determinant of A. Self-affine tiles have rich analytic and number theoretic properties, and have attracted a lot of attention in fractal geometry, wavelet theory and number theory. There are a lot of literature on this topic; see [1,2,13–16] and references therein. Rectangles and regular hexagons are trivial examples of tiles, and the twin dragon is a non-trivial example of tiles.

Among all the properties of $T(A, \mathcal{D})$, the geometric and topological properties are the most important and fundamental, and they therefore are studied most extensively. One of the very interesting aspects is the connectedness of $T(A, \mathcal{D})$. It was asked by [7] that when A is an expansive integer matrix, whether or not there exists a digit set $\mathcal{D}$ such that $\#\mathcal{D} = |\det A|$ and $T(A, \mathcal{D})$ is a connected tile. Various results can be found in [2–5,7,8].

It is almost trivial to see that in $\mathbb{R}$, for $A = q$ an integer larger than or equal to 2, $T(A, \mathcal{D})$ is a connected self-affine tile if and only if $\mathcal{D} = \{\alpha, \alpha + \beta, \cdots, \alpha + (q-1)\beta\}$ for some α and $\beta \neq 0$. [1,9,11,12] considered the connectedness of $T(A, \mathcal{D})$ where $\mathcal{D}$ is a consecutive collinear digit set. [17] explored the case in which $\mathcal{D}$ is a non-consecutive collinear digit set.

Not attempting to study the tiles generated from more general digit sets, [6] considered another simple case of non-collinear digit sets: the matrix A is a 2×2 lower triangular matrix and the digit set is arranged in the form of rectangular, i.e.,

$$A = \begin{bmatrix} p & 0 \\ -a & q \end{bmatrix}, \qquad \mathcal{D} = \left\{ \begin{bmatrix} i \\ j \end{bmatrix} : 0 \le i \le |p| - 1,\ 0 \le j \le |q| - 1 \right\}, \tag{1.4}$$

where a is a real number. A necessary and sufficient condition of the connectedness of $T(A, \mathcal{D})$ is given there. It is interesting to find that the resulting tiles can be disconnected even for very simple matrices and digit sets, e.g., the case $p = q = 2$, $a > 2$. It is observed that the digit sets in [6] have exactly $|p|$ columns corresponding to the left-up element of A. A natural problem is, what will happen if the number of columns is not $|p|$? Motivated by this, we consider the following case: $\mathcal{D}$ has $|q|$ columns and $|p|$ rows.

Throughout the paper, we write

$$E_n = \{0, 1, \cdots, n-1\}, \quad n \ge 1.$$

For the disconnectedness, we have

Theorem 1.1. *Let $p, q \in \mathbb{Z}$ with $|p| > |q| \ge 2$, and let*

$$A = \begin{pmatrix} p & 0 \\ -a & q \end{pmatrix}, \qquad \mathcal{D} = \left\{ (is, jt)^T : i \in E_{|q|},\ j \in E_{|p|} \right\}$$

where $s, t \neq 0$ and a are real numbers. Then $T(A, \mathcal{D})$ is disconnected.

For the connectedness, we have

Theorem 1.2. *Let $p, q \in \mathbb{Z}$ with $3 \le |p| + 1 < |q| < 2|p| - 1$, and let*

$$A = \begin{pmatrix} p & 0 \\ -a & q \end{pmatrix}, \qquad \mathcal{D} = \left\{ (is, jt)^T : i \in E_{|q|},\ j \in E_{|p|} \right\}$$

where $s, t \neq 0$ and a are real numbers. If $q^2 - |qp| \le |\frac{as}{t}| \le \frac{q^2(|p|-1)}{|q|-2}$, then $T(A, \mathcal{D})$ is connected.