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Operators which are the difference of two projections

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ABSTRACT

We study the set $\mathcal{D}$ of differences

$$\mathcal{D} = \{A = P - Q : P, Q \in \mathcal{P}\},$$

where $\mathcal{P}$ denotes the set of orthogonal projections in $\mathcal{H}$. We describe models and factorizations for elements in $\mathcal{D}$, which are related to the geometry of $\mathcal{P}$. The study of $\mathcal{D}$ throws new light on the geodesic structure of $\mathcal{P}$ (we show that two projections in generic position are joined by a unique minimal geodesic). The topology of $\mathcal{D}$ is examined, particularly its connected components are studied. Also we study the subsets $\mathcal{D}_c \subset \mathcal{D}_F$, where $\mathcal{D}_c$ are the compact elements in $\mathcal{D}$, and $\mathcal{D}_F$ are the differences $A = P - Q$ such that the pair (P, Q) is a Fredholm pair ((P, Q) is a Fredholm pair if $QP|_{R(P)} : R(P) \to R(Q)$ is a Fredholm operator).

1. Introduction

We study bounded linear operators in a Hilbert space $\mathcal{H}$ which are the difference of two orthogonal projections:

$$A = P - Q.$$

Such operators A are apparently selfadjoint, and they are contractions. Indeed, by the Krein–Krasnoselski–Milman formula (see for instance [1]),

$$\|P - Q\| = \max\{\|P(1 - Q)\|, \|Q(1 - P)\|\},$$

and clearly $\|P(1 - Q)\| \leq 1$ and $\|Q(1 - P)\| \leq 1$. Also, straightforward computations show that

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$$N(A) = \big(N(P) \cap N(Q)\big) \oplus \big(R(P) \cap R(Q)\big), \qquad N(A-1) = R(P) \cap N(Q)$$

and

$$N(A+1) = N(P) \cap R(Q).$$

Note that $N(A)$, $N(A-1)$, $N(A+1)$, and the orthogonal complement $\mathcal{H}_0$ of the sum of these, reduce P, Q and A. These subspaces depend on A and not on the projections P and Q. The space $\mathcal{H}_0$ is usually called the generic part of P and Q. We shall call it, we guess more appropriately, the generic part of $A = P - Q$. It is the generic part that is of interest, as A acts trivially on the non-generic part. Namely, denote by $A_0 = A|_{\mathcal{H}_0}$ the generic part of A, acting in $\mathcal{H}_0$. Apparently, in the decomposition

$$\mathcal{H} = N(A) \oplus N(A-1) \oplus N(A+1) \oplus \mathcal{H}_0$$

A is given by

$$A = 0 \oplus 1 \oplus -1 \oplus A_0.$$

There is an extensive bibliography on pairs of projections. There is also a very good survey paper on the subject by A. Böttcher and I.M. Spitkovsky [5], and we refer the reader to the references therein. We shall base our remarks on two classic papers on the subject, by P. Halmos [9] and C. Davis [7]. The first of these papers provides a simple 2×2 matrix model for a given pair of projections P, Q, which we describe below. One of the many consequences is that the generic parts P_0 and Q_0 acting in $\mathcal{H}_0$ are unitarily equivalent, with an explicitly constructed unitary operator implementing this equivalence. The second paper characterizes the operators A which are a difference of projections: their generic parts are selfadjoint contractions A_0 which anticommute with a symmetry V (a symmetry is a selfadjoint unitary operator: $V^* = V = V^{-1}$).

We regard the present paper as an incomplete comment on these two papers. Given our interest in the differential geometry of the space $\mathcal{P}$ of projections in $\mathcal{H}$ [6], we relate the results by Halmos and Davis to the question of the existence and uniqueness of geodesics in $\mathcal{P}$.

The contents of the paper are the following. In Section 2 we recall the results by Halmos [9] and Davis [7], as well as certain facts from the geometry of $\mathcal{P}$ [6]. Section 3 contains consequences of Davis' characterization of differences of projections A, particularly, that symmetries V which anticommute with A_0 parametrize all pairs P, Q such that $A = P - Q$. In Section 4 we show how each geodesic of $\mathcal{P}$ joining P and Q provides a factorization $A = e^{iZ}\sigma$, where A, $Z = Z^*$ and $\sigma = \sigma^*$ anticommute (in contrast to the polar decomposition $A = \operatorname{sgn}(A)|A|$, where all data commute). In a previous work [3], it was shown that the projections P_0 and Q_0 in generic position can be joined by a (minimal) geodesic of $\mathcal{P}$. Using the ideas here we show that such geodesic is unique. In Section 5 we obtain descriptions for operators $A = P - Q$ and anticommuting symmetries V, decomposing $\mathcal{H}$ in cyclic subspaces, as in the classic spectral theorem. In Section 6 we examine the topology of the space $\mathcal{D}$ of differences of projections. We study connected components and characterize the interior set of $\mathcal{D}$: it consists of operators A such that A_0 is non-trivial. In Section 7, using results from [4] (also [2]), we study operators $A = P - Q$ such that (P, Q) is a Fredholm pair. From the results obtained in [4] it is apparent that the property of being a Fredholm pair depends on the difference A and not on the particular pair. Therefore, an index for such differences (hereafter referred to as *Fredholm differences*) is defined, which coincides with $\dim(N(A-1)) - \dim(N(A+1))$. This allows us to characterize the connected components of the sets of the Fredholm differences and compact differences, as a consequence.

The main results of the paper are in Theorem 4.2 (factorization of elements in $\mathcal{D}$), Theorem 4.3 and Corollary 4.4 (uniqueness of geodesics joining projections in generic position), Theorem 5.5 (multiplication