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The multifractal spectra for the recurrence rates of beta-transformations

Jung-Chao Ban [a], Bing Li [b,*]

[a] Department of Mathematics, National Dong Hwa University, Hualien 970003, Taiwan
[b] Department of Mathematics, South China University of Technology, Guangzhou 510640, PR China

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ABSTRACT

In this paper, we show a handy approximate approach to provide a lower bound of the Hausdorff dimension of a given subset in $[0, 1)$ related to β-transformation dynamical system. Here approximation means from special class with β-shift satisfying the specification property or being subshift of finite type to general $\beta > 1$. As an application, we obtain the multifractal spectra for the recurrence rate of the first return time of β-transformation, including the cases returning to the ball and cylinder.

1. Introduction

Let $(X, \mathcal{B}, \mu, T, d)$ be a metric measure-preserving system (m.m.p.s.), by which we mean that (X, d) is a metric space, $\mathcal{B}$ is a σ-field containing the Borel σ-field of X and $(X, \mathcal{B}, \mu, T)$ is a measure-preserving dynamical system. Under the assumption that (X, d) has a countable base, Poincaré recurrence theorem implies that μ-almost all $x \in X$ is recurrent in the sense

$$\liminf_{n\to\infty} d\big(T^n x, x\big) = 0 \tag{1.1}$$

(for example, see [11]). Later, Boshernitzan [4] has improved it by a quantitative result

$$\liminf_{n\to\infty} n^{1/\alpha} d\big(T^n x, x\big) < \infty, \quad \mu\text{-almost everywhere (a.e. for short)},$$

where α is the dimension of the space in some sense.

The above results describe whether or not a point is recurrent and how far the orbit will return to the initial point. Recurrence time is an important aspect used to characterize the behaviors of orbits in

* Corresponding author.
E-mail addresses: jcban@mail.ndhu.edu.tw (J.-C. Ban), libing0826@gmail.com (B. Li).

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dynamical systems. Of the research conducted on recurrence time, the first return time of a point has been well studied in the last decade. The first return time of a point $x \in X$ into the set A is defined by

$$\tau_A(x) = \inf\{k \in \mathbb{N} : T^k x \in A\}.$$

Ornstein and Weiss [21] proved that for a finite partition ξ of X, if there exists a T-invariant ergodic Borel probability measure μ, then

$$\lim_{n\to\infty} \frac{\log \tau_{\xi_n(x)}(x)}{n} = h_\mu(\xi), \quad \mu\text{-a.e.}$$

where $\xi_n(x)$ is the intersection of $\xi, T^{-1}(\xi), \cdots, T^{-n+1}(\xi)$ which contains x, and $h_\mu(\xi)$ denotes the measure-theoretic entropy of T with respect to the partition ξ. Feng and Wu [10] considered the recurrence set of the one-sided shift space on m symbols $(\{0,1,\dots,m-1\}^{\mathbb{N}}, \sigma)$, where the partition ξ is the cylinders sets $\{[0],[1],\dots,[m-1]\}$. They proved that the set

$$\left\{x \in \{0,1,\dots,m-1\}^{\mathbb{N}} : \liminf_{n\to\infty} \frac{\log \tau_{\xi_n(x)}(x)}{n} = \alpha,\ \limsup_{n\to\infty} \frac{\log \tau_{\xi_n(x)}(x)}{n} = \gamma\right\}$$

has Hausdorff dimension one for any $0 \le \alpha \le \gamma \le +\infty$ (see also [26]). Lau and Shu [15] extended this result to the dynamical systems with specification property by considering the topological entropy instead of Hausdorff dimension. Barreira and Saussol [2] replaced the cylinders $\xi_n(x)$ with the balls $B(x,r)$ according to quantity

$$\tau_r(x) = \inf\{n \ge 1 : T^n x \in B(x,r)\},$$

and defined the lower and upper recurrence rates of x by

$$\underline{R}(x) = \liminf_{r\to 0} R_r(x), \qquad \overline{R}(x) = \limsup_{r\to 0} R_r(x),$$

where $R_r(x) = \frac{\log \tau_r(x)}{-\log r}$. They proved that

$$\underline{R}(x) = \underline{d}_\mu(x), \qquad \overline{R}(x) = \bar{d}_\mu(x), \quad \mu\text{-a.e.} \tag{1.2}$$

with the conditions that μ has a so-called *long return time* (see [2]) and $\underline{d}_\mu(x) > 0$ for μ-a.e. x, where $\underline{d}_\mu(x)$, $\bar{d}_\mu(x)$ are the lower and upper pointwise dimensions of μ at a point $x \in X$ respectively. A simple consequence of this result is a reformulation of Boshernitzan's theory by noting that

$$\liminf_{n\to\infty} n^{1/\alpha} d(T^n x, x) = 0$$

holds for all $\alpha > \underline{d}_\mu(x)$. Many researchers have studied the problem when the formulation (1.2) holds from many different viewpoints. For example, Saussol [25, Theorem 3] proved that formulation (1.2) holds if the transformation T is piecewise Lipschitz with some condition and the decay of the correlation is super-polynomial.

Let $A(R_r(x))$ be the set of the accumulation points of $R_r(x)$ as $r \to 0$ and J a compact sub-interval of $(0,+\infty)$. Olsen [20] studied the following set

$$G \cap \{x \in K : A(R_r(x)) = J\}$$