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Infinitely many solutions of quasilinear Schrödinger equation with sign-changing potential ☆

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ABSTRACT

In this paper, we study the following quasilinear Schrödinger equation of the form

$$-\Delta u + V(x)u - \Delta\big(u^2\big)u = g(x,u), \quad x \in \mathbb{R}^N,$$

where the potential $V(x)$ is allowed to be sign-changing, and the primitive of the nonlinearity $g(x,u)$ is of superlinear growth at infinity in u and is also allowed to be sign-changing. We obtain the existence of infinitely many nontrivial solutions by using dual approach and Mountain Pass Theorem.

1. Introduction and main results

In this paper, we study the existence and multiplicity of standing wave solutions for the following quasilinear Schrödinger equations of the form

$$-\Delta u + V(x)u - \Delta\big(u^2\big)u = g(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $V \in C(\mathbb{R}^N, \mathbb{R})$ and $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.

This class of quasilinear equations is often referred as modified nonlinear Schrödinger equations whose solutions are related to the existence of standing wave solutions for the following quasilinear Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi + W(x)\psi - \kappa\Delta\big(\rho\big(|\psi|^2\big)\big)\rho'\big(|\psi|^2\big)\psi - g(x,\psi), \quad x \in \mathbb{R}^N, \tag{1.2}$$

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where $W(x)$ is a given potential, κ is a real constant, ρ and g are real functions. We would like to mention that quasilinear equation of the form (1.2) arises in various branches of mathematical physics and has been derived as models of several physical phenomena corresponding to various types of nonlinear term ρ. For instance, the case $\rho(s) = s$ was used to model a superfluid film equation in plasma physics (see Kurihura [12]).

The semilinear case $\kappa = 0$ has been studied extensively in recent years with a huge variety of conditions on the potential V and the nonlinearity g, see for example [1–3,23,28,29,38] and references therein. Compared to the semilinear case, the quasilinear case ($\kappa \neq 0$) becomes much more complicated since the effects of the quasilinear and non-convex term $\Delta(u^2)u$. One of the main difficulties of (1.1) is there is no suitable space on which the energy functional is well defined and belongs to C^1-class except for $N = 1$ (see [21]). To the best of our knowledge, the first existence result involving variational methods due to [21] for $N = 1$ or $V(x)$ is radially symmetrical for high dimensions by using a constrained minimization argument (see also [16] for the more general case). After then, there are some ideas and approaches were developed to overcome the difficulty. See [8,18,24] for a Nehari manifold argument. By using a change of variables (dual approach) the authors in [17] reduced the quasilinear equation to a semilinear one, and an Orlicz space framework was used. The same method was also used in [7], the usual Sobolev space framework was used as the working space. Along this line, there have been a large number of works about existence and multiplicity of problem (1.1), we refer the reader to [4,5,8,9,20,26,27,33,35] and the references therein. But this approach is invalid for the general quasilinear problem (see [14]). Recently, Liu et al. [13,14] developed a perturbation method, the main idea of which is adding a regularizing term to recover the smoothness of the energy functional, so the standard minimax theory can be applied. Soon after, Wu [34] proved the existence of high energy solutions by applying the perturbation method for general quasilinear problem.

Different from the semilinear problems, another feather of the quasilinear problem (1.1) is that the critical exponent is not 2^* but 22^*, where 2^* is the usual Sobolev exponent. The critical problems similar to (1.1) was considered in [4,5,11,15,27,31,37], for more general quasilinear critical problems in [14,19]. For singular perturbation problem and concentration phenomenon of semi-classical states, see, for instance [6,10,11,25, 30,31,36,37] and the references therein.

It is worth pointing out that the aforementioned authors always assumed the potential $V(x)$ is positive. As far as we know there are no papers which deal with the sign-changing potential case for problem (1.1). Motivated by papers [7,17], in the present paper we shall consider problem (1.1) with non-constant and sign-changing potential by a dual approach, and establish the existence of infinitely many large solutions under more general superlinear assumptions on the nonlinearity.

More precisely, we make the following assumptions:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x) > -\infty$;

(V_2) for any $M > 0$, there exists a constant $r > 0$ such that

$$\lim_{|y|\to+\infty} \operatorname{meas}\big(\big\{x \in \mathbb{R}^N : |x - y| \le r, V(x) \le M\big\}\big) = 0;$$

(G_0) $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and there exist constants $c_1, c_2 > 0$ and $4 < p < 22^*$ such that

$$\big|g(x,u)\big| \le c_1|u| + c_2|u|^{p-1}, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R};$$

(G_1) $\lim_{|u|\to\infty} \frac{G(x,u)}{u^4} = \infty$ uniformly in x, and there exists $r_0 \ge 0$ such that $G(x,u) \ge 0$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$ and $|u| \ge r_0$, where $G(x,u) = \int_0^u g(x,s)ds$;

(G_2) $\widetilde{G}(x,u) := \frac{1}{4}g(x,u)u - G(x,u) \ge 0$, and there exist $c_0 > 0$ and $\sigma > \max\{1, \frac{2N}{N+2}\}$ such that

$$\big|G(x,u)\big|^\sigma \le c_0|u|^{2\sigma}\widetilde{G}(x,u) \quad \text{for all } (x,u) \in \mathbb{R}^N \times \mathbb{R} \text{ with } u \text{ large enough;}$$

(G_3) $g(x,u) = -g(x,-u)$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$.