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A Hölder-type inequality on a regular rooted tree

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ABSTRACT

We establish an inequality which involves a non-negative function defined on the vertices of a finite m-ary regular rooted tree. The inequality may be thought of as relating an interaction energy defined on the free vertices of the tree summed over automorphisms of the tree, to a product of sums of powers of the function over vertices at certain levels of the tree. Conjugate powers arise naturally in the inequality, indeed, Hölder's inequality is a key tool in the proof which uses induction on subgroups of the automorphism group of the tree.

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1. Introduction

The energy of an interacting particle system with a finite number of sites is typically given by a sum $\sum_{(\mathbf{i}_1,\mathbf{i}_2)} \mu(\mathbf{i}_1)\mu(\mathbf{i}_2)F(\mathbf{i}_1,\mathbf{i}_2)$ over pairs of particles $(\mathbf{i}_1,\mathbf{i}_2)$ where $F(\mathbf{i}_1,\mathbf{i}_2)$ represents the force or interaction between the particles located at $\mathbf{i}_1$ and $\mathbf{i}_2$ which have masses or weights $\mu(\mathbf{i}_1)$ and $\mu(\mathbf{i}_2)$ respectively.

A more complex system may involve interactions between three or more particles simultaneously rather than just two. Thus we are led to consider energies of the form $\sum_{(\mathbf{i}_1,\dots,\mathbf{i}_n)} \mu(\mathbf{i}_1)\cdots\mu(\mathbf{i}_n)F(\mathbf{i}_1,\dots,\mathbf{i}_n)$ resulting from interactions $F(\mathbf{i}_1,\dots,\mathbf{i}_n)$ which depend on n particles and their configuration. Typically each particle may be affected most by those other particles that are closest, so the interaction should take into account the nearest neighbor structure of the particle configuration. One convenient way of incorporating such a structure is by representing the sites as the free vertices (i.e. vertices of valence 1) of a regular m-ary tree, so that the distance between a pair of sites is an ultrametric determined by the level of their first common ancestor. The arrangement of common ancestors or 'joins' of a collection of n particles determines their nearest neighbor configuration.

Our main results and detailed notation are set out in Section 2, but to fix ideas we give here a brief overview and an example. We work on an m-ary regular rooted tree T of k levels, where the level or generation of a vertex is its edge distance from the root. In the usual parlance, each vertex has m 'children', except for the free vertices at level k which have no children; we let T_0 denote the set of free vertices. We write $\mathbf{i}\wedge\mathbf{i}'$ for the *join* of $\mathbf{i},\mathbf{i}'\in T_0$, that is the vertex $\mathbf{j}\in T$ of maximum level that lies on both of the paths

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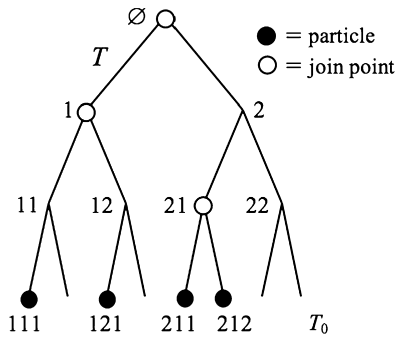

Fig. 1. Particles and join points in the example.

from the root $\emptyset$ to $\mathbf{i}$ and from $\emptyset$ to $\mathbf{i}'$. The *join set* of n particles located at sites $\mathbf{i}_1, \ldots, \mathbf{i}_n \in T_0$, is the set of vertices of T given by $\mathbf{i}_i \wedge \mathbf{i}_j$ for all $1 \le i < j \le n$ (this is made a little more precise in the next section where we allow for multiple join points).

For f a non-negative function defined on the vertices of T, we will consider interactions that can be expressed as the product of the values of f over the join points, that is,

$$F(\mathbf{i}_1, \ldots, \mathbf{i}_n) = f(\mathbf{j}_1) f(\mathbf{j}_2) \cdots f(\mathbf{j}_{n-1})$$

where $\mathbf{j}_1, \mathbf{j}_2, \ldots, \mathbf{j}_{n-1}$ are the join points of $\mathbf{i}_1, \ldots, \mathbf{i}_n$. Then the energy is of the form

$$\sum_{(\mathbf{i}_1,\ldots,\mathbf{i}_n)} \mu(\mathbf{i}_1) \cdots \mu(\mathbf{i}_n) F(\mathbf{i}_1, \ldots, \mathbf{i}_n). \tag{1.1}$$

Typically, $f(\mathbf{j})$ will be large if the join point $\mathbf{j}$ is a high level vertex corresponding to a significant interaction component resulting from nearby particles. For many problems one needs to bound the energy of a system, perhaps in the limit as the number of generations becomes large. Such estimates are required, for example, in estimating high moments of certain measures, see [2,3].

Whilst one may wish to estimate the sum (1.1) over all arrangements of n particles, the sum breaks down naturally into sub-sums over configurations of particles which have isomorphic join structures. Thus we consider the set of configurations of n particles on T_0 that may be obtained from each other under some automorphism of the rooted tree, in other words the equivalence classes of configurations defined by the automorphisms. These equivalence classes are the orbits of the automorphism group of T acting on the ordered n-tuples from T_0. Writing $[I]$ for such an equivalence class or orbit, we are led to consider the sums

$$\sum_{(\mathbf{i}_1,\ldots,\mathbf{i}_n) \in [I]} \mu(\mathbf{i}_1) \cdots \mu(\mathbf{i}_n) F(\mathbf{i}_1, \ldots, \mathbf{i}_n).$$

We will obtain upper bounds for these sums over the equivalence class $[I]$ in terms of pth powers of f summed across certain levels of the tree T.

To illustrate this, consider a specific case of the binary rooted tree of $k = 3$ levels with $n = 4$ particles. Starting with the configuration $(111, 121, 211, 212)$ of points of T_0 (with the usual coding of vertices of the binary tree) the join points are at $\emptyset$, 1 and 21, see Fig. 1. Let $[I]$ be the class of 64 different equivalent ordered 4-tuples $(g(111), g(121), g(211), g(212))$ obtainable under an automorphism g of the rooted tree T. Assign each free vertex $\mathbf{i} \in T_0$ a positive weight $\mu(\mathbf{i})$. For each vertex $\mathbf{j} \in T$ write $\mu(\mathbf{j})$ for the total weight of the free vertices below $\mathbf{j}$, so, for example, $\mu(21) = \mu(211) + \mu(212)$. In this special case, our main inequality becomes, for any non-negative function f on the vertices of T and for all $p_1, p_2, p_3 > 2$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$,

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