J. Math. Anal. Appl. 423 (2015) 924–939

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Sets of ideal convergence of sequences of quasi-continuous functions

Tomasz Natkaniec *, Jolanta Wesołowska

Institute of Mathematics, University of Gdańsk, Wita Stwosza 57, 80-952 Gdańsk, Poland

A R T I C L E   I N F O

Article history:
Received 5 July 2014
Available online 16 October 2014
Submitted by M. Laczkovich

Keywords:
Quasi-continuous function
Ideal convergence
F_σ-ideal
ω-+-Diagonalizable ideal
κ-Generated ideal
Weakly Ramsey ideal

A B S T R A C T

Lunina's 7-tuples $(E^1, \dots, E^7)$ of sets of pointwise convergence, divergence to ∞, divergence to $-\infty$, etc., for sequences of quasi-continuous functions are characterized. Generalizations on ideal convergence are discussed.

1. Introduction

Let $\vec{f} = (f_n)_n$ be a sequence of real-valued continuous functions defined on a metric space X. It is not difficult to show that the set $C(\vec{f})$ of all $x \in X$ such that $(f_n(x))_n$ converges is $F_{\sigma\delta}$. On the other hand, Hahn [7] and Sierpiński [18] proved independently that for every $F_{\sigma\delta}$ set $A \subset X$ there is a sequence $\vec{f}$ of continuous functions for which $A = C(\vec{f})$. Further research (see e.g. Kornfel'd [9] and Lipiński [11,12]) involved also sets of points where the sequence is divergent to infinity and the like. The full description of these sets was given by Lunina [13], see Theorem 2 below. We will expand this result in two directions. Firstly, the class of all continuous functions will be replaced by the class $\mathrm{QC}(X)$ of all quasi-continuous functions, see Theorem 3. This result will generalize Wesołowska's Theorem on Lipiński's triples for sequences of quasi-continuous functions, see Corollary 4. The second kind of extension of Lunina's Theorem consists in substitution of pointwise convergence of $\vec{f}$ by ideal convergence with respect to some ideal $\mathcal{I}$ on the natural numbers. Borzestowski and Recław proved in [2] an ideal version of Lunina's Theorem for sequences of

* Corresponding author.
E-mail addresses: mattn@mat.ug.edu.pl (T. Natkaniec), jolwesol@pg.gda.pl (J. Wesołowska).

http://dx.doi.org/10.1016/j.jmaa.2014.10.029
0022-247X/

continuous functions and for F_σ ideals. Using analogous methods we will prove ideal versions of Corollary 4 and Theorem 3.

2. Preliminaries

2.1. Notations

Let X be a metric space. For $A \subset X$ we denote by $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{fr} A$ the interior, closure and boundary of A, respectively. $B(x,\varepsilon)$ denotes the open ball centred at x and with the radius ε. A set A is *semi-open* in X if $A \subset \operatorname{cl}(\operatorname{int} A)$. A family of all semi-open sets in X is denoted by $\mathcal{SO}(X)$. The symbols $\mathcal{M}(X)$ and $\mathit{Baire}(X)$ denote the families of all meager sets and all sets with the Baire property in X. We will write $\mathcal{SO}$, $\mathcal{M}$ and Baire instead of $\mathcal{SO}(X)$, $\mathcal{M}(X)$ and $\mathit{Baire}(X)$ if X is fixed. The cardinality of a set A is denoted by $|A|$. The cardinals $\operatorname{add}(\mathcal{M}(X)) = \min\{|\mathcal{A}|: \mathcal{A} \subset \mathcal{M}(X)\ \&\ \bigcup\mathcal{A} \notin \mathcal{M}(X)\}$ and $\operatorname{cov}(\mathcal{M}(X)) = \min\{|\mathcal{A}|: \mathcal{A} \subset \mathcal{M}(X)\ \&\ X = \bigcup\mathcal{A}\}$ are called *additivity* and *covering* of category in X. Recall that $\omega < \operatorname{add}(\mathcal{M}(X)) \le \operatorname{cov}(\mathcal{M}(X)) \le \mathfrak{c}$ for every uncountable Polish space X and those cardinals can be different in different models of ZFC.

For a function $f: X \to \mathbb{R}$ let $C(f)$ denote the set of all continuity points of f and $D(f) = X \setminus C(f)$. By $\mathrm{C}(X)$ we denote the class of all continuous real-valued functions defined on X. We will write C instead of $\mathrm{C}(X)$, if X is fixed.

2.2. Quasi-continuous functions

We will consider the following weak form of continuity which has been introduced by Kempisty [8]. We say that a function $f: X \to \mathbb{R}$ is *quasi-continuous* at a point $x_0 \in X$ if for each open set $U \ni x_0$ and each open set $V \ni f(x_0)$ there is a non-empty open set $W \subset U$ with $f(W) \subset V$. f is quasi-continuous if it is quasi-continuous at each point $x \in X$. The class of all quasi-continuous real-valued functions defined on a space X is denoted by $\mathrm{QC}(X)$ (or QC if X is fixed). It is known that $f: X \to \mathbb{R}$ is quasi-continuous iff $f^{-1}(U)$ is semi-open for each open set $U \subset \mathbb{R}$. Clearly $\mathcal{SO}(X) \subset \mathit{Baire}(X)$, thus every quasi-continuous function has the Baire property. Note that if X is a Baire space then for any quasi-continuous function $f: X \to \mathbb{R}$ the set $C(f)$ is dense (hence residual) in X. Recall also that the sum of a continuous function and a quasi-continuous one is quasi-continuous. (See, e.g. [1].) More properties of quasi-continuous functions can be found e.g. in [17].

In constructions of quasi-continuous functions we apply the following lemma.

Lemma 1 *(Borsik). (See [1, Lemma 1].) Let X be an arbitrary metric space. For every nonempty nowhere dense closed set $F \subset X$ satisfying $F \subset \operatorname{cl} G$ for some nonempty open set $G \subset X$ there exists a collection $\{K_{n,m}: n \in \mathbb{N}, m \le n\}$ of nonempty open sets such that:*

(1) *$\operatorname{cl} K_{n,m} \subset G \setminus F$ for all $n \in \mathbb{N}$ and $m \le n$;*
(2) *$\operatorname{cl} K_{n,m} \cap \operatorname{cl} K_{i,j} = \emptyset$ for $(n,m) \ne (i,j)$;*
(3) *for every $x \notin F$ there exists an open neighbourhood V of x such that the set $\{(n,m): \operatorname{cl} K_{n,m} \cap V \ne \emptyset\}$ has at most one element;*
(4) *for every $x \in F$, its open neighbourhood V and a number $m \in \mathbb{N}$ there exists $n \ge m$ such that $K_{n,m} \cap V \ne \emptyset$.*

Consequently, $F \subset \operatorname{cl} \bigcup_{n \ge m} K_{n,m}$ for each $m \in \mathbb{N}$ and both $F \cup \bigcup_{n \ge m} \operatorname{cl} K_{n,m}$ and $F \cup \bigcup_{n \in \mathbb{N}, m \le n} \operatorname{cl} K_{n,m}$ are closed in G.