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Decay of solutions of a second order differential equation with non-smooth second member

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A B S T R A C T

Consider the following abstract initial value problem

$$(*)\quad \begin{cases} u''(t) + \mu(t)Au(t) + a\big(\big|A^{-\frac{\theta}{2}}u(t)\big|^2\big)u(t) + b\big(\big|A^{-\eta}u'(t)\big|\big)Au'(t) = f(t) \\ \quad \text{in } (0,\infty); \\ u(0) = u^0, \quad u'(0) = u^1 \end{cases}$$

in a real separable Hilbert space H with norm $|u|$. Here A is a positive self-adjoint operator of H; $\mu(t), a(s), b(s)$ positive functions, $f(t)$ a vectorial non-smooth function and θ, η real numbers. In this paper we study the existence, uniqueness and decay of solutions of problem $(*)$. In our approach, we use the Theory of Self-Adjoint Operators in Hilbert spaces, the compactness Aubin–Lions Theorem and a Lyapunov functional.

1. Introduction

Let V and H be two real separable Hilbert spaces, V continuously embedding in H. The scalar product and norms of H and V, are denoted, respectively, by (u,v), $|u|$ and $((u,v))$, $\|u\|$. Let A be the self-adjoint operator defined by the triplet $\{V, H, ((u,v))\}$. Consider $\alpha \geq 0$, $\alpha \in \mathbb{R}$. Then

$$D(A) = \big\{u \in H; A^\alpha u \in H\big\}$$

is a Hilbert space equipped with the scalar product

$$(u,v)_{D(A^\alpha)} = \big(A^\alpha u, A^\alpha v\big)$$

cf. J.L. Lions [7].

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In the above conditions, we have the following initial value problem:

$$(*)\quad \left|\begin{array}{l} u''(t)+\mu(t)Au(t)+a\big(\big|A^{-\frac{\theta}{2}}u(t)\big|^2\big)u(t)+b\big(\big|A^{-\eta}u'(t)\big|\big)Au'(t)=f(t) \quad \text{in } (0,\infty); \\ u(0)=u^0, \quad u'(0)=u^1. \end{array}\right.$$

Here $\mu(t)$, $a(s)$, $b(s)$ are positive functions; θ, η real numbers and $f(t) \in D(A^\alpha)'$, $D(A^\alpha)'$ dual of $D(A^\alpha)$.

We motivate problem $(*)$. Grotta Ragazzo [2] studied the equation

$$u_{tt}-u_{xx}-au+\left(\frac{1}{\pi}\int_0^\pi u^2dx\right)^\alpha u=0 \quad \text{in } (0,\pi)\times\mathbb{R} \tag{1.1}$$

as an approximation of the Klein–Gordon equation

$$u_{tt}-u_{xx}-au+u^{1+2\alpha}=0 \quad \text{in } (0,\pi)\times\mathbb{R}. \tag{1.2}$$

Observe that Eq. (1.2) with $\alpha = 1$ and $a = 0$ is the meson equation of Schiff [12] (cf. also Jörgens [3]). Louredo, Araújo and Milla Miranda [9] analyzed the equation

$$u''-\mu(t)\triangle u+a\bigg(\int_\Omega u^2dx\bigg)u+b\bigg(\int_\Omega u'^2dx\bigg)u'=0 \quad \text{in } \Omega\times(0,\infty) \tag{1.3}$$

with a nonlinear boundary condition. Here Ω is a bounded domain of $\mathbb{R}^n$. The physical motivation of Eq. (1.3) when Ω is an open interval can be found in [9].

J.L. Lions [6] formulated an open question on the existence of solutions of the equation

$$y''-\triangle y+\left(\int_0^t\int_\Omega y^2dxds\right)y=v(t)\delta(x-b) \quad \text{in } \Omega\times(0,\infty) \tag{1.4}$$

where $\delta(x-b)$ is the Dirac mass supported at $\{b\}$, $b\in\Omega$.

In Milla Miranda, Louredo and Medeiros [11] an answer to this question with a modification of the nonlinear term is given. Eq. (1.4) motivates us to consider the non-smooth function f in Eq. $(*)$.

Existence of solutions u of problem $(*)$ with $b = 0$ and a particular $a(s)$ was obtained in Milla Miranda, Louredo and Medeiros [11]. The term $b(|A^{-\eta}u'|)Au'$ is introduced in Eq. $(*)$ with the finality of obtaining the decay of solutions of the problem.

In this paper the existence of bounded solutions of problem $(*)$ and the uniqueness of solutions for particular real numbers θ and η are obtained. The exponential decay of the energy associated to $(*)$ is also derived. In Section 5 we give some applications of our result. In our approach, in the existence of solutions, we use the Theory of Self-Adjoint Operators in Hilbert spaces and the compactness Aubin–Lions Theorem. In the decay of solutions, a functional of Lyapunov is applied.

2. Main results

We use the notation $D(A^\alpha)'=D(A^{-\alpha})$, $\alpha\in\mathbb{R}$, $\alpha\geq 0$. Identifying H with H', we have

$$D(A^\alpha)\hookrightarrow H\hookrightarrow D(A^{-\alpha}).$$

Here and in what follows the notation $X\hookrightarrow Y$ means that the space X is dense in the space Y and the embedding of X in Y is continuous. Note that $D(A^{-\alpha})'=D(A^\alpha)$. Also, if $\beta,\gamma\in\mathbb{R}$ with $\beta\geq\gamma$, we have

$$D(A^\beta)\hookrightarrow D(A^\gamma).$$